

Local-in-time Well-posedness of Boundary Layer System for the Full Incompressible MHD Equations by Energy Methods

Jincheng Gao Boling Guo Daiwen Huang

Institute of Applied Physics and Computational Mathematics,
100088, Beijing, P. R. China

Abstract

In this paper, we investigate the well-posedness theory for the MHD boundary layer system in two-dimensional space. The boundary layer equations are governed by the Prandtl type equations that are derived from the full incompressible MHD system with non-slip boundary condition on the velocity, perfectly conducting condition on the magnetic field, and Dirichlet boundary condition on the temperature when the viscosity coefficient depends on the temperature. To derive the Prandtl type boundary layer system, we require all the hydrodynamic Reynolds numbers, magnetic Reynolds numbers and Nusselt numbers tend to infinity at the same rate. Under the assumption that the initial tangential magnetic field is not zero, one applies the energy methods to establish the local-in-time existence and uniqueness of solution for the MHD boundary layer equations without the necessity of monotonicity condition.

Keywords: Prandtl type equations, Full incompressible MHD equations, Well-posedness, Sobolev space, non-monotone condition.

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1 Introduction

The dynamics of an electrically conducting liquid near a wall has been a topic of constant interest since the pioneering work of Hartmann [1]. An appropriate starting point to describe such dynamics is the classical incompressible magnetohydrodynamics(MHD) system. One important problem about MHD is to understand the high Reynolds and Nusselt numbers limit in a domain with boundary. In this paper, we investigate the following initial boundary value problem for the two dimensional full incompressible MHD system in a periodic domain $\Omega = \{(x, y) : x \in \mathbb{T}, y \in \mathbb{R}^+\}$:

$$\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \varepsilon \operatorname{div}(2\mu(\vartheta^\varepsilon) D(u^\varepsilon)) + \nabla p^\varepsilon = (H^\varepsilon \cdot \nabla) H^\varepsilon, \\ c_v [\partial_t \vartheta^\varepsilon + (u^\varepsilon \cdot \nabla) \vartheta^\varepsilon] - \varepsilon \kappa \Delta \vartheta^\varepsilon = 2\varepsilon \mu(\vartheta^\varepsilon) |D(u^\varepsilon)|^2 + \varepsilon \nu |\nabla \times H^\varepsilon|^2, \\ \partial_t H^\varepsilon - \nabla \times (u^\varepsilon \times H^\varepsilon) - \varepsilon \nu \Delta H^\varepsilon = 0, \\ \operatorname{div} u^\varepsilon = 0, \quad \operatorname{div} H^\varepsilon = 0, \end{cases} \quad (1.1)$$

The unknown function $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ denotes the velocity vector, $H^\varepsilon = (h_1^\varepsilon, h_2^\varepsilon)$ denotes the magnetic field, ϑ^ε denotes the absolute temperature, and $p^\varepsilon = \tilde{p}^\varepsilon + \frac{1}{2}|H^\varepsilon|^2$ represents the total pressure with \tilde{p}^ε the pressure of fluid. Here, $\mu(\vartheta^\varepsilon)$ means that μ is a smooth function of temperature ϑ^ε , and $\varepsilon \mu(\vartheta^\varepsilon), \varepsilon \kappa$ and

$\varepsilon\nu$ represent the viscosity, heat conductivity and resistivity coefficients respectively. To obtain the same boundary layer thickness, we assume the viscosity, heat conductivity and resistivity coefficients have the the same order of a small parameter ε . The positive constant c_v is the heat capacity coefficient, and the deformation tensor $D(u^\varepsilon)$ is defined by

$$D(u^\varepsilon) = \frac{1}{2} [\nabla u^\varepsilon + (\nabla u^\varepsilon)^{tr}].$$

To complete the system (1.1), the boundary conditions are given by

$$u^\varepsilon|_{y=0} = 0, \quad \partial_y h_1^\varepsilon|_{y=0} = h_2^\varepsilon|_{y=0} = 0, \quad \vartheta^\varepsilon|_{y=0} = 0. \quad (1.2)$$

As the parameter ε tends to zero in the systems (1.1), we obtain the following systems formally

$$\begin{cases} \partial_t u^0 + (u^0 \cdot \nabla) u^0 + \nabla P^0 = (H^0 \cdot \nabla) H^0, \\ \partial_t \vartheta^0 + (u^0 \cdot \nabla) \vartheta^0 = 0, \\ \partial_t H^0 - \nabla \times (u^0 \times H^0) = 0, \\ \operatorname{div} u^0 = 0, \quad \operatorname{div} H^0 = 0, \end{cases} \quad (1.3)$$

which are the ideal MHD systems with energy equation. Then, it is easy to check that there is a mismatch of boundary condition between the equations (1.1) and (1.3) on the boundary $y = 0$, which will form the boundary layer as in the vanishing viscosity, heat conductivity and resistivity limit process. To find out the terms in (1.1) whose contributions is essential for the boundary layer, we use the same scaling as the one used in [2],

$$t = t, \quad x = x, \quad \tilde{y} = \varepsilon^{-\frac{1}{2}} y,$$

then set

$$\begin{aligned} u_1(t, x, \tilde{y}) &= u_1^\varepsilon(t, x, y), & u_2(t, x, \tilde{y}) &= \varepsilon^{-\frac{1}{2}} u_2^\varepsilon(t, x, y), \\ h_1(t, x, \tilde{y}) &= h_1^\varepsilon(t, x, y), & h_2(t, x, \tilde{y}) &= \varepsilon^{-\frac{1}{2}} h_2^\varepsilon(t, x, y), \end{aligned}$$

and

$$\theta(t, x, \tilde{y}) = \vartheta^\varepsilon(t, x, y), \quad p(t, x, \tilde{y}) = p^\varepsilon(t, x, y).$$

In this paper, we assume the viscosity function $\mu(\vartheta^\varepsilon)$ has the following form

$$\mu(\vartheta^\varepsilon) \triangleq \mu \vartheta^\varepsilon + \mu. \quad (1.4)$$

Then by taking the leading order, we deduce from the equations (1.1) that

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 - \mu \partial_y [(\vartheta + 1) \partial_y u_1] + \partial_x p = h_1 \partial_x h_1 + h_2 \partial_y h_1, \\ \partial_y p = 0, \\ c_v (\partial_t \vartheta + u_1 \partial_x \vartheta + u_2 \partial_y \vartheta) - \kappa \partial_y^2 \vartheta = \mu (\vartheta + 1) (\partial_y u_1)^2 + \nu (\partial_y h_1)^2, \\ \partial_t h_1 + \partial_y (u_2 h_1 - u_1 h_2) - \nu \partial_y^2 h_1 = 0, \\ \partial_t h_2 - \partial_x (u_2 h_1 - u_1 h_2) - \nu \partial_y^2 h_2 = 0, \\ \partial_x u_1 + \partial_y u_2 = 0, \quad \partial_x h_1 + \partial_y h_2 = 0, \end{cases} \quad (1.5)$$

where $(t, x, y) \in [0, T] \times \Omega$, here we have replaced \tilde{y} by y for simplicity of notations. Indeed, the nonlinear boundary layer systems (1.5) become the classical well-known unsteady boundary layer systems if the magnetic field vanishes, refer to [3].

The second equation of equations (1.5)₂ implies that the leading order of boundary layers for the total pressure $p^\varepsilon(t, x, y)$ is invariant across the boundary layer, and should be matched to the outflow pressure $P(t, x)$ on top of boundary layer, that is, the trace of pressure of idea MHD flow. Hence, we obtain

$$p(t, x, y) \equiv P(t, x).$$

Furthermore, the tangential component $u_1(t, x, y)$ of velocity field, $h_1(t, x, y)$ of magnetic field, temperature $\vartheta(t, x, y)$, should match the outflow tangential velocity $U(t, x)$, outflow tangential magnetic field $H(t, x)$ and the outflow temperature $\Theta(t, x)$, on the top of boundary layer, that is

$$u_1(t, x, y) \rightarrow U(t, x), \quad h_1(t, x, y) \rightarrow H(t, x), \quad \vartheta(t, x, y) \rightarrow \Theta(t, x), \quad \text{as } y \rightarrow +\infty, \quad (1.6)$$

where $U(x, t)$, $H(x, t)$ and $\Theta(x, t)$ are the trace of tangential velocity, tangential magnetic field and temperature respectively. Then, we have the following matching conditions:

$$\partial_t U + U \partial_x U + \partial_x P = H \partial_x H, \quad \partial_t \Theta + U \partial_x \Theta = 0, \quad \partial_t H + U \partial_x H - H \partial_x U = 0. \quad (1.7)$$

Moreover, by virtue of (1.2), one attains the following boundary conditions

$$u_1|_{y=0} = u_2|_{y=0} = \vartheta|_{y=0} = \partial_y h_1|_{y=0} = h_2|_{y=0} = 0. \quad (1.8)$$

On the other hand, it is noted that the equation (1.5)₅ is a direct consequences of equations (1.5)₄, (1.5)₆ and the boundary conditions (1.8). Hence, we only need to study the following initial boundary value problem for the nonlinear MHD boundary layer equations

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 - \mu \partial_y [(\vartheta + 1) \partial_y u_1] + P_x = h_1 \partial_x h_1 + h_2 \partial_y h_1, \\ c_v (\partial_t \vartheta + u_1 \partial_x \vartheta + u_2 \partial_y \vartheta) - \kappa \partial_y^2 \vartheta = \mu (\vartheta + 1) (\partial_y u_1)^2 + \nu (\partial_y h_1)^2, \\ \partial_t h_1 + \partial_y (u_2 h_1 - u_1 h_2) - \nu \partial_y^2 h_1 = 0, \\ \partial_x u_1 + \partial_y u_2 = 0, \quad \partial_x h_1 + \partial_y h_2 = 0, \end{cases} \quad (1.9)$$

with the boundary conditions

$$(u_1, u_2, \vartheta, \partial_y h_1, h_2)(t, x, y)|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} (u_1, \vartheta, h_1)(t, x, y) = (U, \Theta, H)(t, x). \quad (1.10)$$

and the initial data

$$u_1(t, x, y)|_{t=0} = u_{10}(x, y), \quad \vartheta(t, x, y)|_{t=0} = \vartheta_0(x, y), \quad h_1(t, x, y)|_{t=0} = h_{10}(x, y). \quad (1.11)$$

Let us first introduce some weighted Sobolev spaces for later use. Denote

$$\Omega \triangleq \{(x, y) : x \in \mathbb{T}, y \in \mathbb{R}^+\}.$$

For any $l \in \mathbb{R}$, denote by $L_l^2(\Omega)$ the weighted Lebesgue space with respect to the spatial variables:

$$L_l^2(\Omega) \triangleq \{f(x, y) : \Omega \rightarrow \mathbb{R}, \|f\|_{L_l^2(\Omega)} \triangleq \left(\int_{\Omega} \langle y \rangle^{2l} |f(x, y)|^2 dx dy \right)^{\frac{1}{2}} < +\infty\}, \quad \langle y \rangle \triangleq 1 + y,$$

and then, for any given $m \in \mathbb{N}$, denote by $H_l^m(\Omega)$ the weighted Sobolev space:

$$H_l^m(\Omega) \triangleq \{f(x, y) : \Omega \rightarrow \mathbb{R}, \|f\|_{H_l^m(\Omega)} \triangleq \left(\sum_{m_1+m_2 \leq m} \|\langle y \rangle^{l+m_2} \partial_x^{m_1} \partial_y^{m_2} f\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} < +\infty\}.$$

Now, we can state the main results with respect to the well-posedness theory for the nonlinear MHD boundary layer systems (1.9) in this paper as follows.

Theorem 1.1. *Let $m \geq 5$ be an integer, and $l \geq 0$ be a real number. Assume that the outer flow $(U, \Theta, H, P_x)(t, x)$ satisfies that for some $T > 0$,*

$$M_0 \triangleq \sum_{i=0}^{2m+2} \sup_{0 \leq t \leq T} \|\partial_t^i(U, \Theta, H, P)(t)\|_{H^{2m+2-i}(\mathbb{T}_x)} < +\infty, \quad (1.12)$$

and $\Theta(t, x) \geq 0$ for all $(t, x) \in [0, T] \times \mathbb{T}_x$. Also, we suppose the initial data $(u_{10}, \vartheta_0, h_{10})(x, y)$ satisfies

$$\vartheta_0(x, y) \geq 0, \quad (u_{10}(x, y) - U(0, x), \vartheta_0(x, y) - \Theta(0, x), h_{10}(x, y) - H(0, x)) \in H_l^{3m+2}(\Omega), \quad (1.13)$$

and the compatibility conditions up to m -th order. Moreover, there exists a sufficiently small constant $\delta_0 > 0$ such that

$$h_{10}(x, y) \geq 2\delta_0, \quad |\langle y \rangle^{l+1} \partial_y^i(u_{10}, \vartheta_0, h_{10})(x, y)| \leq (2\delta_0)^{-1}, \quad \text{for } i = 1, 2, \quad (x, y) \in \Omega. \quad (1.14)$$

Then, there exist a positive time $0 < T^ \leq T$ and a unique solution $(u_1, u_2, \vartheta, h_1, h_2)$ to the initial boundary value problem (1.9), such that*

$$(u_1 - U, \vartheta - \Theta, h_1 - H) \in \bigcap_{i=0}^m W^{i,\infty}(0, T_*; H_l^{m-i}(\Omega)), \quad (1.15)$$

and

$$\begin{aligned} (u_2 + U_x y, h_2 + H_x y) &\in \bigcap_{i=0}^{m-1} W^{i,\infty}(0, T_*; H_{-1}^{m-1-i}(\Omega)), \\ (\partial_y u_2 + U_x, \partial_y h_2 + H_x) &\in \bigcap_{i=0}^{m-1} W^{i,\infty}(0, T_*; H_l^{m-1-i}(\Omega)). \end{aligned} \quad (1.16)$$

We now review some related works to the problem studied in this paper. The vanishing viscosity limit of the incompressible Navier-Stokes equations that, in a bounded domain with Dirichlet boundary condition, is an important problem in both physics and mathematics. This is due to the formation of a boundary layer, where the solution undergoes a sharp transition from a solution of the Euler system to the zero non-slip boundary condition on boundary of the Navier-Stokes system. This boundary layer satisfies the Prandtl system formally. Indeed, Prandtl [4] derived the Prandtl equations for boundary layers from the incompressible Navier-Stokes equations with non-slip boundary condition. The first systematic work in rigorous mathematics was obtained Oleinik [5], in which she established the local in time well-posedness of the Prandtl equations in dimension two by applying the Crocco transformation under the monotonicity condition on the tangential velocity field in the normal direction to the boundary. For more extensional mathematical results, the interested readers can refer to the classical book finished by Oleinik and Samokhin [2]. By taking care of the cancelation in the convection term to overcome the loss of derivative in the tangential direction of velocity, the researchers in [6] and [7] independently used the simply energy method to establish well-posedness theory for the two-dimensional Prandtl equations in the framework of Sobolev spaces. Moreover, Xin and Zhang [8] built the global in time weak solution by imposing an additional favorable condition on the pressure. Furthermore, the well-posedness results for both classical and weak solutions in dimension three were studied by Liu et al.[9, 10]. On the other hand, Sammartino and Caffisch [11, 12] obtained the well-posedness in the framework of analytic functions without the monotonicity condition on the velocity field and justified the boundary layer expansion. For more results to the Prandtl equations in the framework of analytic functions, the interested readers can

refer to [13–18] and the references therein. And recently, the analyticity condition can be further relaxed to Gevrey regularity, cf. [19–22].

When the monotonicity condition is violated, separation of the boundary layer is expected and observed for classical fluid. Hence, E and Engquist [23] constructed a finite time blowup solution to the Prandtl system for some special type of initial data. Recently, Gérard-Varet and Dormy [24] proved ill-posedness for the linearized Prandtl equations around a nonmonotonic shear flow. For more interesting ill-posedness(or instability) phenomena of solution to both the linear and nonlinear Prandtl equations around the shear flow, the readers can refer to [25–32] and the references therein. All these results show that the monotonicity assumption on the tangential velocity is essential for the well-posedness except in the framework of analytic functions or Gevrey functions. On the other hand, as observed by Van Dommelen and Shen [33] and studied mathematically by Hong and Hunter [34], the monotonicity condition is not needed for the well-posedness of the inviscid Prandtl equations at least locally in time. Recently, the well-posedness of thermal layer equations, which was derived from the full compressible Navier-Stokes equations when the viscosity coefficients vanish or are of higher order with respect to the heat conductivity coefficient, were obtained by Liu et al.[35] without the monotonicity condition on the velocity field in dimension three.

Under the influence of electro-magnetic field, the system of magnetohydrodynamics(denoted by MHD) is a fundamental system to describe the movement of electrically conducting fluid, for example plasmas and liquid metals, refer to [36]. For plasma, the boundary layer equations, which can be derived from the fundamental MHD system, are more complicated than the classical Prandtl system because of the coupling of the magnetic field with velocity field through the Maxwell equations. If the magnetic field is transversal to the boundary, there are extensive discussions on the so-called Hartmann boundary layer, refer to [37, 38]. In addition, there are works on the stability of boundary layers with minimum Reynolds number for flow with different structure to reveal the difference from the classical boundary layers electro-magnetic field, refer to [39–41]. Under the non-slip boundary condition for the velocity, the well-posedness theory for the boundary layer systems, which were derived if the hydrodynamic Reynolds numbers tend to infinity while the magnetic Reynolds numbers are fixed, was discussed in Oleinik and Samokhin [2], for which the monotonicity condition on the velocity field is needed. However, if both the hydrodynamic Reynolds numbers and magnetic Reynolds numbers tend to infinity at the same rate, the local-in-time existence of solution for the boundary layer system was obtained by Liu et al.[42] under the only condition on the initial tangential magnetic field was not zero. It should be pointed out that the well-posedness for this boundary layer system does not need the monotonicity condition of tangential velocity. At the same time, Gérard-Varet and Prestipino [43] provided a systematic derivation of boundary layer models in magnetohydrodynamics, through an asymptotic analysis of the incompressible MHD system. Furthermore, they also performed some stability analysis for the boundary layer system, and emphasized the stabilizing effect of the magnetic field.

In this paper, we derive the boundary layer systems (1.9) by requiring all the hydrodynamic Reynolds numbers, magnetic Reynolds numbers and Nusselt numbers tend to infinity at the same rate. On one hand, it is believed that the magnetic field has a stabilizing effect on the boundary layer that could provide a mechanism for containment of the high temperature gas in physics. On the other hand, Liu et al.[42] established the local well-posedness theory for the MHD boundary layer systems(without energy equations) under the only condition on the initial tangential magnetic field was not zero. Hence, the prime objective of this paper is to prove the local existence and uniqueness for the two dimensional MHD boundary layer systems with temperature field. Now, let us explain the main difficulties arising from the appearance of temperature field as well as the our strategies for overcoming them. First of all, we should

establish the lower bound estimate for the temperature field to give $L^2(0, T; \mathcal{H}_t^m)$ -norm for the quantity $\partial_y u$ because the viscosity coefficient depends on the temperature field. Due to the lack of viscous term $\partial_x^2 \vartheta$, we can't apply the minimum principle to attain the lower bound estimate for the temperature field. Hence, we assume the viscosity function obeys the form(see (1.4)), which will help us reach the target by means of energy method. Secondly, the lack of high-order boundary conditions at $y = 0$ prevent us from applying the integration by parts in the y -variable, but it will be solvable by taking the operator $\partial_t - \partial_y^2$ since the viscosity coefficient has a good form(see (1.4)). Thirdly, some higher order nonlinear terms arising in the energy equation will bring some difficulties when we apply the energy method to establish local well-posedness theory for the MHD boundary layer systems. However, we can choose the life span of solutions small suitably to overcome these difficulties since we only investigate the local existence of solutions in this paper. Finally, similar to the classical Prandtl equations, the convective term $u_2 \partial_y \vartheta$ in the energy equation (1.9)₂ will create a loss of x -derivative estimate. Indeed, we can take the strategy of cancelation property and create a quantity θ_β (see (3.94)) to avoid the x -derivative estimate of temperature field.

The rest of this paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, one establishes the a priori energy estimates for the nonlinear problem (1.9). The local-in-time existence and uniqueness of the solution to (1.9) in Sobolev space are given in Section 4. Finally, some useful inequalities and important equivalent relations will be stated in Appendixs A and B.

2 Preliminaries

First of all, we introduce some notations which will be used frequently in this paper. Denote the tangential derivative operator

$$\partial_\tau^\beta = \partial_t^{\beta_1} \partial_x^{\beta_2}, \quad \text{for } \beta = (\beta_1, \beta_2) \in \mathbb{N}^2, \quad |\beta| = \beta_1 + \beta_2,$$

and then denote the derivative operator(in both time and space) by

$$D^\alpha = \partial_\tau^\beta \partial_y^k, \quad \text{for } \alpha = (\beta_1, \beta_2, k) \in \mathbb{N}^3, \quad |\alpha| = |\beta| + k.$$

Set $e_i \in \mathbb{N}^2, i = 1, 2$, and $E_j \in \mathbb{N}^3, j = 1, 2, 3$, by

$$e_1 = (1, 0), \quad e_2 = (0, 1), \quad E_1 = (1, 0, 0), \quad E_2 = (0, 1, 0), \quad E_3 = (0, 0, 1),$$

and denote by ∂_y^{-1} the inverse of derivative ∂_y , i.e., $(\partial_y^{-1} f)(y) \triangleq \int_0^y f(z) dz$. Furthermore, the notation $[\cdot, \cdot]$ denotes the commutator operator, and $\mathcal{P}(\cdot)$ represents a nondecreasing polynomial function that may differ from line to line. For any integer m , define the function space \mathcal{H}_t^m of measurable functions $f(t, x, y) : [0, T] \times \Omega \rightarrow \mathbb{R}$, such that for any $t \in [0, T]$,

$$\|f(t)\|_{\mathcal{H}_t^m} \triangleq \left(\sum_{|\alpha| \leq m} \|\langle y \rangle^{k+l} D^\alpha f(t, \cdot)\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} < +\infty. \quad (2.1)$$

Similar to Liu et al.[42], we introduce an auxiliary function $\phi(y)$ satisfying that

$$\phi(y) = \begin{cases} y, & y \geq 2R_0, \\ 0, & 0 \leq y \leq R_0, \end{cases} \quad (2.2)$$

which will help us overcome the technical difficulty originated from the boundary terms at $y = +\infty$. Then, set the new unknown functions:

$$\begin{aligned} u(t, x, y) &\triangleq u_1(t, x, y) - U(t, x)\phi'(y), & v(t, x, y) &\triangleq u_2(t, x, y) + U_x(t, x)\phi(y), \\ h(t, x, y) &\triangleq h_1(t, x, y) - H(t, x)\phi'(y), & g(t, x, y) &\triangleq h_2(t, x, y) + H_x(t, x)\phi(y), \\ \theta(t, x, y) &\triangleq \vartheta(t, x, y) - \Theta(t, x)\phi'(y). \end{aligned} \quad (2.3)$$

Choose the above construction for (u, v, θ, h, g) to ensure the divergence free conditions:

$$\partial_x u + \partial_y v = 0, \quad \partial_x h + \partial_y g = 0,$$

and homogeneous boundary conditions:

$$(u, v, \theta, \partial_y h, g)|_{y=0} = 0, \quad \lim_{y \rightarrow \infty} (u, \theta, h)(t, x, y) = 0.$$

Then, it is easy to check that

$$v = -\partial_y^{-1} \partial_x u, \quad g = -\partial_y^{-1} \partial_x h. \quad (2.4)$$

At the same time, one can deduce from the relation (2.3) that

$$u = (u_1 - U) + U(1 - \phi'(y)), \quad \theta = (\vartheta - \Theta) + \Theta(1 - \phi'(y)), \quad h = (h_1 - H) + H(1 - \phi'(y)),$$

which, together with the construction of $\phi(y)$ (see the definition in (2.2)), yields immediately

$$\|(u, \theta, h)(t)\|_{\mathcal{H}_t^m} - CM_0 \leq \|(u_1 - U, \vartheta - \Theta, h_1 - H)(t)\|_{\mathcal{H}_t^m} \leq \|(u, \theta, h)(t)\|_{\mathcal{H}_t^m} + CM_0, \quad (2.5)$$

where the quantity M_0 is defined in (1.12). By using the new unknown function (u, v, θ, h, g) , we can reformulate the original problem (1.9) as the following form

$$\left\{ \begin{aligned} &\partial_t u + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h \\ &\quad - \mu\partial_y[(\theta + \Theta\phi'(y) + 1)\partial_y u] + U_x\phi'u + U\phi''v - H_x\phi'h - H\phi''g - U\phi^{(3)}\theta - U\phi''\theta_y = r_1, \\ &c_v\{\partial_t\theta + (u + U\phi')\partial_x\theta + (v - U_x\phi)\partial_y\theta\} - \kappa\partial_y^2\theta + c_v\Theta_x\phi'u + c_v\Theta\phi''v - \mu\theta(u_y)^2 - \mu(U\phi'')^2\theta \\ &\quad - 2\mu U\phi''\theta u_y - \mu\Theta\phi'(u_y)^2 - 2\mu\Theta U\phi'\phi''u_y - 2\mu U\phi''u_y - \mu(u_y)^2 - \nu(h_y)^2 - 2\nu H\phi'h_y = r_2, \\ &\partial_t h + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u - \nu\partial_y^2 h \\ &\quad + H_x\phi'u + H\phi''v - U_x\phi'h - U\phi''g = r_3, \\ &\partial_x u + \partial_y v = 0, \quad \partial_x h + \partial_y g = 0, \end{aligned} \right. \quad (2.6)$$

with the boundary and initial conditions

$$(u, v, \theta, \partial_y h, g)(t, x, y)|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} (u, \theta, h)(t, x, y) = 0, \quad (u, \theta, h)(t, x, y)|_{t=0} = (u_0, \theta_0, h_0)(x, y), \quad (2.7)$$

where

$$\left\{ \begin{aligned} r_1 &\triangleq U_t[(\phi')^2 - \phi' - \phi\phi''] + P_x[(\phi')^2 - \phi\phi'' - 1] + U\Theta[\phi'\phi^{(3)} + (\phi'')^2] + U\phi'\phi^{(3)}, \\ r_2 &\triangleq c_v\Theta_t[(\phi')^2 - \phi'] + c_vU_x\Theta\phi\phi'' + \kappa\Theta\phi^{(3)} - \mu\Theta\phi'(U\phi'')^2 + \mu(U\phi'')^2 + \kappa(\Theta\phi'')^2 + \nu(H\phi'')^2, \\ r_3 &\triangleq H_t[(\phi')^2 - \phi' + \phi\phi''] + \nu H\phi^{(3)}, \quad u_0 \triangleq u_{10}(x, y) - U(0, x)\phi'(y), \\ \theta_0(x, y) &\triangleq \vartheta_0(x, y) - \Theta(0, x)\phi'(y), \quad h_0(x, y) \triangleq h_{10}(x, y) - H(0, x)\phi'(y). \end{aligned} \right. \quad (2.8)$$

In view of the definition $\phi(y)$, it is easy to check that

$$\begin{aligned} r_1(t, x, y), \quad r_2(t, x, y), \quad r_3(t, x, y) &= 0, \quad y \geq 2R_0, \\ r_1(t, x, y) &= -P_x(t, x), \quad r_2(t, x, y) = r_3(t, x, y) = 0, \quad 0 \leq y \leq R_0, \end{aligned} \quad (2.9)$$

and for any $t \in [0, T]$, $\lambda \geq 0$ and $|\alpha| \leq m$, then one gets that

$$\|\langle y \rangle^\lambda D^\alpha(r_1, r_2, r_3)(t)\|_{L^2(\Omega)} \leq C \sum_{|\beta| \leq |\alpha|+1} \|\partial_\tau^\beta(U, \Theta, H, P_x)\|_{L^2(\mathbb{T}_x)}^2 \leq CM_0^2. \quad (2.10)$$

Furthermore, we have the following relation for the initial data

$$\|(u_0, \theta_0, h_0)\|_{\mathcal{H}_l^m} - CM_0 \leq \|(u_{10} - U(0, x), \vartheta_0 - \Theta(0, x), h_{10} - H(0, x))(t)\|_{\mathcal{H}_l^m} \leq \|(u_0, \theta_0, h_0)\|_{\mathcal{H}_l^m} + CM_0. \quad (2.11)$$

Finally, from the transform (2.3) and the relation (2.4), it is easy to know that the well-posedness theory in Theorem 1.1 is a corollary of the following result.

Theorem 2.1. *Let $m \geq 5$ be an integer, $l \geq 0$ be a real number, and $(U, \Theta, H, P_x)(t, x)$ satisfies the hypotheses given in Theorem 1.1. In addition, assume that for the problem (2.6), the initial data satisfies $\theta_0(x, y) + \Theta(0, x)\phi'(y) \geq 0$, $(u_0(x, y), \theta_0(x, y), h_0(x, y)) \in H_l^{3m+2}(\Omega)$, and the compatibility condition up to m -th order. Moreover, there exists a sufficiently small constant $\delta_0 > 0$, such that*

$$h_0(x, y) + H(0, x)\phi'(y) \geq 2\delta_0, \quad |\langle y \rangle^{l+1} \partial_y^i(u_0, \theta_0, h_0)(x, y)| \leq (2\delta_0)^{-1}, \quad \text{for } i = 1, 2, (x, y) \in \Omega. \quad (2.12)$$

Then, there exists a time $0 < T_* \leq T$ and a unique solution $(u, v, \theta, h, g)(t, x, y)$ to the initial boundary value problem (2.6), such that

$$(u, \theta, h) \in \bigcap_{i=0}^m W^{i, \infty}(0, T_*; H_l^{m-i}(\Omega)), \quad (2.13)$$

and

$$(v, g) \in \bigcap_{i=0}^{m-1} W^{i, \infty}(0, T_*; H_{-1}^{m-1-i}(\Omega)), \quad (\partial_y v, \partial_y g) \in \bigcap_{i=0}^{m-1} W^{i, \infty}(0, T_*; H_l^{m-1-i}(\Omega)). \quad (2.14)$$

Therefore, our main task is to show the local-posedness theory in the above Theorem 2.1, and its proof will be given in the following two sections.

3 A priori estimates

In this section, we will establish a priori estimates for the nonlinear MHD boundary layer problem (2.6).

Proposition 3.1. *[Weighted estimates for $D^m(u, \theta, h)$]*

Let $m \geq 5$ be an integer, $l \geq 0$ be a real number, and the hypotheses for $(U, \Theta, H, P_x)(t, x)$ given in Theorem 1.1 hold. Assume that (u, v, θ, h, g) is a classical solution to the problem (2.6) in $[0, T]$, satisfying that $(u, \theta, h) \in L^\infty(0, T; \mathcal{H}_l^m)$, $(\partial_y u, \partial_y \theta, \partial_y h) \in L^2(0, T; \mathcal{H}_l^m)$, and for sufficiently small δ_0 :

$$h(t, x, y) + H(t, x)\phi'(y) \geq \delta_0, \quad \langle y \rangle^{l+1} \partial_y^i(u, \theta, h)(t, x, y) \leq \delta_0^{-1}, \quad i = 1, 2, \quad (t, x, y) \in [0, T] \times \Omega. \quad (3.1)$$

Then, it holds that for small time,

$$\begin{aligned} \sup_{0 \leq s \leq t} \|(u, \theta, h)(s)\|_{\mathcal{H}_l^m}^2 &\leq \delta_0^{-4} \{ \mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_l^{2m}(\Omega)}) + CM_0^{10} t \}^{\frac{1}{2}} \\ &\cdot \left\{ 1 - C\delta_0^{-48} t (\mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_l^{2m}(\Omega)}) + CM_0^{10} t)^5 \right\}^{-\frac{1}{10}}. \end{aligned} \quad (3.2)$$

Also, we have that for $i = 1, 2$,

$$\begin{aligned}
 & \|\langle y \rangle^{l+1} \partial_y^i(u, \theta, h)(t, x, y)\|_{L^\infty(\Omega)} \\
 & \leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, \theta_0, h_0)(x, y)\|_{L^\infty(\Omega)} \\
 & \quad + C\delta_0^{-4}t \left\{ \mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_l^{2m}(\Omega)}) + CM_0^{10}t \right\}^{\frac{1}{2}} \\
 & \quad \cdot \left\{ 1 - C\delta_0^{-48}t(\mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_l^{2m}(\Omega)}) + CM_0^{10}t)^5 \right\}^{-\frac{1}{10}}.
 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
 h(t, x, y) & \geq h_0(x, y) - C\delta_0^{-4}t \left\{ \mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_l^{2m}(\Omega)}) + CM_0^{10}t \right\}^{\frac{1}{2}} \\
 & \quad \cdot \left\{ 1 - C\delta_0^{-48}t(\mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_l^{2m}(\Omega)}) + CM_0^{10}t)^5 \right\}^{-\frac{1}{10}}.
 \end{aligned} \tag{3.4}$$

Here $\mathcal{P}(\cdot)$ denotes a nondecreasing polynomial function.

3.1. Lower bound estimate for temperature

In this subsection, we obtain lower bound estimate for the temperature field, which will play an important role in giving $L^2(0, T; \mathcal{H}_l^m)$ -norm for the quantity $\partial_y u$.

Lemma 3.2. *Let $m \geq 5$ be an integer, $l \geq 0$ be a real number, and the hypotheses for $(U, \Theta, H, P_x)(t, x)$ given in Theorem 1.1 hold on. Assume that (u, v, θ, g, h) is a classical solution to the nonlinear problem (2.6) in $[0, T]$, and satisfies $(u, \theta, h) \in L^\infty(0, T; \mathcal{H}_l^m)$, $(\partial_y u, \partial_y \theta, \partial_y h) \in L^2(0, T; \mathcal{H}_l^m)$. Then, it holds that*

$$\theta(t, x, y) + \Theta(t, x)\phi'(y) \geq 0, \tag{3.5}$$

for almost everywhere $(t, x, y) \in [0, T] \times \mathbb{T}_x \times \mathbb{R}^+$.

Proof. By virtue of $\lim_{y \rightarrow +\infty} \vartheta(t, x, y) = \Theta(t, x)$, for all $(x, t) \in \mathbb{T}_x \times [0, T]$. Then there exists a positive constant R_1 such that for $y \geq R_1$, we have

$$\vartheta(t, x, y) - \Theta(t, x) \geq -\frac{1}{2} \inf_{(t, x)} \Theta(t, x),$$

which implies

$$\vartheta(t, x, y) \geq \Theta(t, x) - \frac{1}{2} \inf_{(t, x)} \Theta(t, x) \geq \frac{1}{2} \inf_{(t, x)} \Theta(t, x), \tag{3.6}$$

for all $(t, x, y) \in [0, T] \times \mathbb{T}_x \times [R_1, +\infty)$. Let $l_1 \triangleq \min_{(x, y) \in \mathbb{T}_x \times [0, R_1+1]} \vartheta_0(x, y) \geq 0$, $l_2 \triangleq \frac{1}{2} \inf_{(t, x)} \Theta(t, x) \geq 0$, $l \triangleq \min\{l_1, l_2, 0\}$, and $\Omega_0 \triangleq \mathbb{T}_x \times [0, R_1 + 1]$. For any $0 < \varepsilon_0 < 1$, let $k \triangleq l - \varepsilon_0$, multiplying (1.9)₂ by $(k - \vartheta)_+$ and integrating the resulting equality over $[0, t] \times \Omega_0$, we find

$$\begin{aligned}
 & \frac{c_v}{2} \int_0^t \frac{d}{d\tau} \int_{\Omega_0} |(k - \vartheta)_+|^2 dx dy d\tau + \kappa \int_0^t \int_{\Omega_0} |\partial_y(k - \vartheta)_+|^2 dx dy d\tau \\
 & = \mu \int_0^t \int_{\Omega_0} (\partial_y u_1)^2 |(k - \vartheta)_+|^2 dx dy d\tau - \mu(k+1) \int_0^t \int_{\Omega_0} (\partial_y u_1)^2 (k - \vartheta)_+ dx dy d\tau \\
 & \quad - \nu \int_0^t \int_{\Omega_0} (\partial_y h_1)^2 (k - \vartheta)_+ dx dy d\tau,
 \end{aligned} \tag{3.7}$$

here $f_+ \triangleq \max\{f, 0\} \geq 0$. It is easy to deduce from the (3.7) that

$$\int_{\Omega_0} |(k - \vartheta)_+(t)|^2 dx dy \leq 2\mu c_v^{-1} \int_0^t \|\partial_y u_1\|_{L^\infty(\Omega)}^2 \int_{\Omega_0} |(k - \vartheta)_+|^2 dx dy d\tau. \quad (3.8)$$

Then, the application of the Grönwall inequality to (3.8) yields immediately

$$(k - \vartheta)_+(x, y, t) = 0, \quad \text{a.e. } (t, x, y) \in [0, T] \times \Omega_0,$$

which, implies that

$$\vartheta(t, x, y) \geq k = l - \varepsilon_0, \quad \text{a.e. } (t, x, y) \in [0, T] \times \Omega_0. \quad (3.9)$$

Let $\varepsilon_0 \rightarrow 0^+$ in (3.9), then it is easy to deduce that

$$\vartheta(t, x, y) \geq l, \quad \text{a.e. } (t, x, y) \in [0, T] \times \Omega_0,$$

which, together with (3.6), yields $\vartheta(t, x, y) \geq 0$, for all $(t, x, y) \in [0, T] \times \mathbb{T}_x \times \mathbb{R}^+$. Then, the construction of function (2.3) helps us complete the proof of Lemma 3.2. \square

Remark 3.1. The equation (1.9)₂ is not a standard parabolic type equation due to the lack of viscous term $\partial_x^2 \vartheta$. Then, we can't apply the minimum principle to obtain the estimate (3.5) for temperature field. In order to reach the target (3.5), we assume the viscosity function $\mu(\vartheta^\varepsilon)$ obeys the form (1.4).

3.2. Weighted H_l^m - estimates with norm derivatives

For any $|\alpha| = |\beta| + k \leq m$ and $|\beta| \leq m - 1$, the weighted estimates on $D^\alpha(u, \theta, h)$ can be obtained by the standard energy method since one order regularity loss is allowed. Then, we can obtain the following estimates:

Lemma 3.3. [Weighted estimates for $D^\alpha(u, \theta, h)$ with $|\alpha| \leq m, |\beta| \leq m - 1$]

Let $m \geq 5$ be an integer, $l \geq 0$ be a real number, and the hypotheses for $(U, \Theta, H, P_x)(t, x)$ given in Theorem 1.1 hold. Assume that $(u, v, \theta, g, h)(t, x, y)$ is a classical solution to the problem (2.6) in $[0, T]$, and satisfies $(u, \theta, h)(t, x, y) \in L^\infty(0, T; \mathcal{H}_l^m)$, $(\partial_y u, \partial_y \theta, \partial_y h)(t, x, y) \in L^2(0, T; \mathcal{H}_l^m)$. Then, there exists a positive constant C , depending on m, l and ϕ such that for any small $0 < \delta_1 < 1$

$$\begin{aligned} & \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \left(\frac{d}{dt} \|D^\alpha(u, \sqrt{c_v} \theta, h)(t)\|_{L_{k+l}^2(\Omega)}^2 + c_0 \|(D^\alpha \partial_y u, D^\alpha \partial_y \theta, D^\alpha \partial_y h)(t)\|_{L_{k+l}^2(\Omega)}^2 \right) \\ & \leq \delta_1 \|(\partial_y u, \partial_y \theta, \partial_y h)(t)\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} (\|(u, \theta, h)(t)\|_{\mathcal{H}_0^m}^8 + 1) \\ & \quad + \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1, r_2, r_3)(t)\|_{L_{l+k}^2(\Omega)}^2 + C \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, \Theta, H, P)(t)\|_{L^2(\mathbb{T}_x)}^8, \end{aligned} \quad (3.10)$$

where $c_0 \triangleq \min\{\mu, \kappa, \nu\}$.

Proof. Applying the operator $D^\alpha = \partial_\tau^\beta \partial_y^k$ for $\alpha = (\beta, k) = (\beta_1, \beta_2, k)$, satisfying $|\alpha| = |\beta| + k \leq m, |\beta| \leq m - 1$, to the equations (2.6)₁, (2.6)₂ and (2.6)₃ respectively, it yields that

$$\begin{cases} \partial_t D^\alpha u = D^\alpha r_1 + \mu \partial_y D^\alpha [(\theta + \Theta \phi' + 1) \partial_y u] - D^\alpha I_1, \\ c_v \partial_t D^\alpha \theta = D^\alpha r_2 + \kappa \partial_y^2 D^\alpha \theta - D^\alpha I_2, \\ \partial_t D^\alpha h = D^\alpha r_3 + \nu \partial_y^2 D^\alpha h - D^\alpha I_3, \end{cases} \quad (3.11)$$

where the functions $I_i (i = 1, 2, 3)$ are defined as follows:

$$\begin{cases} I_1 \triangleq [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h \\ \quad + U_x\phi'u + U\phi''v - H_x\phi'h - H\phi''g - U\phi^{(3)}\theta - U\phi''\theta_y; \\ I_2 \triangleq c_v[(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\theta + c_v\Theta_x\phi'u + c_v\Theta\phi''v - \mu\theta(u_y)^2 - \mu(U\phi'')^2\theta \\ \quad - 2\mu U\phi''\theta u_y - \mu\Theta\phi'(u_y)^2 - 2\mu\Theta U\phi'\phi''u_y - 2\mu U\phi''u_y - \mu(u_y)^2 - \nu(h_y)^2 - 2\nu H\phi'h_y; \\ I_3 \triangleq [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u + H_x\phi'u \\ \quad + H\phi''v - U_x\phi'h - U\phi''g. \end{cases}$$

Multiplying (3.11)₁ by $\langle y \rangle^{2l+2k} D^\alpha u$, (3.11)₂ by $\langle y \rangle^{2l+2k} D^\alpha \theta$, and (3.11)₃ by $\langle y \rangle^{2l+2k} D^\alpha h$ respectively, and integrating them over Ω with respect to the spatial variables x and y , we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \langle y \rangle^{2k+2l} |D^\alpha u|^2 dx dy + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \langle y \rangle^{2k+2l} |D^\alpha h|^2 dx dy + \frac{c_v}{2} \frac{d}{dt} \int_{\Omega} \langle y \rangle^{2k+2l} |D^\alpha \theta|^2 dx dy \\ &= \int_{\Omega} (D^\alpha r_1 \cdot \langle y \rangle^{2k+2l} D^\alpha u + D^\alpha r_2 \cdot \langle y \rangle^{2k+2l} D^\alpha \theta + D^\alpha r_3 \cdot \langle y \rangle^{2k+2l} D^\alpha h) dx dy \\ &\quad - \int_{\Omega} (D^\alpha I_1 \cdot \langle y \rangle^{2k+2l} D^\alpha u + D^\alpha I_2 \cdot \langle y \rangle^{2k+2l} D^\alpha \theta + D^\alpha I_3 \cdot \langle y \rangle^{2k+2l} D^\alpha h) dx dy \\ &\quad + \mu \int_{\Omega} \partial_y D^\alpha [(\theta + \Theta\phi' + 1)\partial_y u] \cdot \langle y \rangle^{2k+2l} D^\alpha u dx dy + \kappa \int_{\Omega} \partial_y^2 D^\alpha \theta \cdot \langle y \rangle^{2k+2l} D^\alpha \theta dx dy \\ &\quad + \nu \int_{\Omega} \partial_y^2 D^\alpha h \cdot \langle y \rangle^{2k+2l} D^\alpha h dx dy. \end{aligned} \tag{3.12}$$

First of all, the application of Cauchy inequality implies immediately

$$\begin{aligned} & \int_{\Omega} (D^\alpha r_1 \cdot \langle y \rangle^{2k+2l} D^\alpha u + D^\alpha r_2 \cdot \langle y \rangle^{2k+2l} D^\alpha \theta + D^\alpha r_3 \cdot \langle y \rangle^{2k+2l} D^\alpha h) dx dy \\ & \leq \frac{1}{2} \|D^\alpha(u, \theta, h)\|_{L_{l+k}^2(\Omega)}^2 + \frac{1}{2} \|D^\alpha(r_1, r_2, r_3)\|_{L_{l+k}^2(\Omega)}^2. \end{aligned} \tag{3.13}$$

Next, we assume the following two estimates hold, which will be proved later: for any small $0 < \delta_1 < 1$,

$$\begin{aligned} & \mu \int_{\Omega} \partial_y D^\alpha [(\theta + \Theta\phi' + 1)\partial_y u] \cdot \langle y \rangle^{2k+2l} D^\alpha u dx dy \\ & \quad + \kappa \int_{\Omega} \partial_y^2 D^\alpha \theta \cdot \langle y \rangle^{2k+2l} D^\alpha \theta dx dy + \nu \int_{\Omega} \partial_y^2 D^\alpha h \cdot \langle y \rangle^{2k+2l} D^\alpha h dx dy \\ & \leq -\frac{\mu}{2} \|D^\alpha \partial_y u\|_{L_{k+l}^2(\Omega)}^2 - \frac{\kappa}{2} \|D^\alpha \partial_y \theta\|_{L_{k+l}^2(\Omega)}^2 - \frac{\nu}{2} \|D^\alpha \partial_y h\|_{L_{k+l}^2(\Omega)}^2 + \delta_1 \mu \|\partial_y u\|_{\mathcal{H}_0^m}^2 \\ & \quad + \delta_1 \kappa \|\partial_y \theta\|_{\mathcal{H}_0^m}^2 + \delta_1 \nu \|\partial_y h\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1} (\|(u, \theta, h)\|_{\mathcal{H}_0^m}^6 + 1) + \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(\Theta, P)\|_{L^2(\mathbb{T}_x)}^4, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} & - \int_{\Omega} (D^\alpha I_1 \cdot \langle y \rangle^{2k+2l} D^\alpha u + D^\alpha I_2 \cdot \langle y \rangle^{2k+2l} D^\alpha \theta + D^\alpha I_3 \cdot \langle y \rangle^{2k+2l} D^\alpha h) dx dy \\ & \leq \frac{\mu}{4} \|D^\alpha \partial_y u\|_{L_{l+k}^2(\Omega)}^2 + \frac{\nu}{4} \|D^\alpha \partial_y h\|_{L_{l+k}^2(\Omega)}^2 + \frac{\kappa}{4} \|D^\alpha \partial_y \theta\|_{L_{k+l}^2(\Omega)}^2 + \delta_1 \mu \|\partial_y u\|_{\mathcal{H}_0^m}^2 \\ & \quad + \delta_1 \nu \|\partial_y h\|_{\mathcal{H}_0^m}^2 + \delta_1^{-1} (\|(u, \theta, h)\|_{\mathcal{H}_l^m}^8 + 1) + \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, \Theta, H)\|_{L^2(\mathbb{T}_x)}^8. \end{aligned} \tag{3.15}$$

By plugging the estimates (3.13)-(3.15) into (3.12), one obtains

$$\begin{aligned} & \frac{d}{dt} \|D^\alpha(u, \sqrt{c_v}\theta, h)(t)\|_{L_{k+l}^2(\Omega)}^2 + \frac{1}{4} \min\{\mu, \kappa, \nu\} \|D^\alpha \partial_y(u, \theta, h)(t)\|_{L_{k+l}^2(\Omega)}^2 \\ & \leq \delta_1 \max\{\mu, \nu, \kappa\} \|(\partial_y u, \partial_y \theta, \partial_y h)(t)\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1} (\|(u, \theta, h)(t)\|_{\mathcal{H}_t^m}^8 + 1) \\ & \quad + \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1, r_2, r_3)(t)\|_{L_{l+k}^2(\Omega)}^2 + \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, \Theta, H, P)(t)\|_{L^2(\mathbb{T}_x)}^8. \end{aligned} \quad (3.16)$$

which implies the estimate (3.10) immediately. Therefore, we complete the proof of Lemma 3.3. \square

Proof of (3.14). In this part, we will first handle the term $\mu \int_\Omega \partial_y D^\alpha[(\theta + \Theta\phi' + 1)\partial_y u] \cdot \langle y \rangle^{2k+2l} D^\alpha u dx dy$. By integration by parts, we have

$$\begin{aligned} & \mu \int_\Omega \partial_y D^\alpha[(\theta + \Theta\phi' + 1)\partial_y u] \cdot \langle y \rangle^{2k+2l} D^\alpha u dx dy \\ & = -\mu \int_{\mathbb{T}_x} D^\alpha[(\theta + \Theta\phi' + 1)\partial_y u] \cdot D^\alpha u|_{y=0} dx \\ & \quad - 2(k+l)\mu \int_\Omega D^\alpha[(\theta + \Theta\phi' + 1)\partial_y u] \cdot \langle y \rangle^{2k+2l-1} D^\alpha u dx dy \\ & \quad - \mu \int_\Omega D^\alpha[(\theta + \Theta\phi' + 1)\partial_y u] \cdot \langle y \rangle^{2k+2l} \partial_y D^\alpha u dx dy \\ & = -\mu \int_\Omega (\theta + \Theta\phi' + 1) \langle y \rangle^{2k+2l} |\partial_y D^\alpha u|^2 dx dy \\ & \quad - \mu \int_\Omega [D^\alpha, (\theta + \Theta\phi' + 1)] \partial_y u \cdot \langle y \rangle^{2k+2l} D^\alpha \partial_y u dx dy \\ & \quad - 2(k+l)\mu \int_\Omega D^\alpha[(\theta + \Theta\phi' + 1)\partial_y u] \cdot \langle y \rangle^{2k+2l-1} D^\alpha u dx dy \\ & \quad - \mu \int_{\mathbb{T}_x} D^\alpha[(\theta + 1)\partial_y u] \cdot D^\alpha u|_{y=0} dx. \end{aligned} \quad (3.17)$$

It is complicated to deal with four terms on the righthand side of (3.17), then we estimate them by the following four steps.

Step 1: In view of the lower boundedness of ϑ in (3.5), it is easy to deduce that

$$\theta + \Theta\phi' + 1 \geq 1,$$

which, implies directly that

$$-\mu \int_\Omega (\theta + \Theta\phi' + 1) \langle y \rangle^{2k+2l} |\partial_y D^\alpha u|^2 dx dy \leq -\mu \int_\Omega \langle y \rangle^{2k+2l} |\partial_y D^\alpha u|^2 dx dy. \quad (3.18)$$

Step 2: By virtue of the Hölder inequality, one deduces that

$$\begin{aligned} & \left| \mu \int_\Omega [D^\alpha, (\theta + \Theta\phi' + 1)] \partial_y u \cdot \langle y \rangle^{2k+2l} D^\alpha \partial_y u dx dy \right| \\ & \leq \mu \sum_{0 < \tilde{\alpha} \leq \alpha} \|D^{\tilde{\alpha}}(\theta + \Theta\phi') \cdot D^{\alpha-\tilde{\alpha}} \partial_y u\|_{L_{k+l}^2(\Omega)} \|D^\alpha \partial_y u\|_{L_{k+l}^2(\Omega)}. \end{aligned} \quad (3.19)$$

For any $m \geq 4$, we apply the inequality (A.3) and Cauchy inequality to get

$$\begin{aligned}
 & \|D^{\tilde{\alpha}}\theta \cdot D^{\alpha-\tilde{\alpha}}\partial_y u\|_{L^2_{k+l}(\Omega)} \|D^{\alpha}\partial_y u\|_{L^2_{k+l}(\Omega)} \\
 & \leq \|D^{\tilde{\alpha}-E_i} D^{E_i}\theta \cdot D^{\alpha-\tilde{\alpha}}\partial_y u\|_{L^2_{k+l}(\Omega)} \|D^{\alpha}\partial_y u\|_{L^2_{k+l}(\Omega)} \\
 & \leq \|D^{E_i}\theta\|_{\mathcal{H}_0^{m-1}} \|\partial_y u\|_{\mathcal{H}_{l+1}^{m-1}} \|D^{\alpha}\partial_y u\|_{L^2_{k+l}(\Omega)} \\
 & \leq \frac{1}{12} \|D^{\alpha}\partial_y u\|_{L^2_{k+l}(\Omega)}^2 + C\|\theta\|_{\mathcal{H}_0^m}^2 \|u\|_{\mathcal{H}_l^m}^2,
 \end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
 & \|D^{\tilde{\alpha}}(\Theta\phi') \cdot D^{\alpha-\tilde{\alpha}}\partial_y u\|_{L^2_{k+l}(\Omega)} \|D^{\alpha}\partial_y u\|_{L^2_{k+l}(\Omega)} \\
 & \leq \|\partial_{\tau}^{\tilde{\beta}}\Theta\|_{L^{\infty}(\mathbb{T}_x)} \|D^{\alpha-\tilde{\alpha}}\partial_y u\|_{L^2_{k+l}(\Omega)} \|D^{\alpha}\partial_y u\|_{L^2_{k+l}(\Omega)} \\
 & \leq \frac{1}{12} \|D^{\alpha}\partial_y u\|_{L^2_{k+l}(\Omega)}^2 + C\|\partial_{\tau}^{\tilde{\beta}}\Theta\|_{H^1(\mathbb{T}_x)}^2 \|u\|_{\mathcal{H}_l^m}^2.
 \end{aligned} \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.19), one arrives at for $m \geq 4$ that

$$\begin{aligned}
 & \left| -\mu \int_{\Omega} [D^{\alpha}, (\theta + \Theta\phi' + 1)] \partial_y u \cdot \langle y \rangle^{2k+2l} D^{\alpha} \partial_y u dx dy \right| \\
 & \leq \frac{\mu}{6} \|D^{\alpha}\partial_y u(t)\|_{L^2_{k+l}(\Omega)}^2 + C\|(u, \theta)(t)\|_{\mathcal{H}_l^m}^4 + C \sum_{|\beta| \leq m+1} \|\partial_{\tau}^{\beta}\Theta(t)\|_{L^2(\mathbb{T}_x)}^4.
 \end{aligned} \tag{3.22}$$

Step 3: By virtue of the Hölder inequality, it is easy to check that

$$\begin{aligned}
 & \left| -2(k+l)\mu \int_{\Omega} D^{\alpha}[(\theta + \Theta\phi' + 1)\partial_y u] \cdot \langle y \rangle^{2k+2l-1} D^{\alpha} u dx dy \right| \\
 & \leq 2(k+l)\mu \|\partial_y D^{\alpha} u\|_{L^2_{k+l}(\Omega)} \|(\theta + \Theta\phi' + 1)D^{\alpha} u\|_{L^2_{k+l-1}(\Omega)} \\
 & \quad + 2(k+l)\mu \sum_{0 < \tilde{\alpha} \leq \alpha} \|D^{\tilde{\alpha}}(\theta + \Theta\phi') \cdot D^{\alpha-\tilde{\alpha}}u_y\|_{L^2_{k+l-1}(\Omega)} \|D^{\alpha} u\|_{L^2_{k+l}(\Omega)}.
 \end{aligned} \tag{3.23}$$

One applies the Sobolev and Cauchy inequalities to obtain

$$\begin{aligned}
 & 2(k+l)\mu \|\partial_y D^{\alpha} u\|_{L^2_{k+l}(\Omega)} \|(\theta + \Theta\phi' + 1)D^{\alpha} u\|_{L^2_{k+l-1}(\Omega)} \\
 & \leq 2(k+l)\mu \|\partial_y D^{\alpha} u\|_{L^2_{k+l}(\Omega)} (\|\theta\|_{L^{\infty}(\Omega)} + \|\Theta\|_{L^{\infty}(\mathbb{T}_x)} + 1) \|D^{\alpha} u\|_{L^2_{k+l-1}(\Omega)} \\
 & \leq \frac{\mu}{12} \|\partial_y D^{\alpha} u\|_{L^2_{k+l}(\Omega)}^2 + C(\|\theta\|_{H^2(\Omega)} + \|\Theta\|_{H^1(\mathbb{T}_x)} + 1)^2 \|D^{\alpha} u\|_{L^2_{k+l-1}(\Omega)}^2.
 \end{aligned} \tag{3.24}$$

By virtue of the inequality (A.3) and Sobolev inequality, it is easy to deduce that

$$\begin{aligned}
 & \|D^{\tilde{\alpha}}\theta \cdot D^{\alpha-\tilde{\alpha}}\partial_y u\|_{L^2_{k+l-1}(\Omega)} \|D^{\alpha} u\|_{L^2_{k+l}(\Omega)} \\
 & = \|D^{\tilde{\alpha}-E_i} D^{E_i}\theta \cdot D^{\alpha-\tilde{\alpha}}\partial_y u\|_{L^2_{k+l-1}(\Omega)} \|D^{\alpha} u\|_{L^2_{k+l}(\Omega)} \\
 & \leq \|\theta\|_{\mathcal{H}_0^m} \|\partial_y u\|_{\mathcal{H}_l^{m-1}} \|D^{\alpha} u\|_{L^2_{k+l}(\Omega)} \\
 & \leq C\|\theta\|_{\mathcal{H}_0^m}^2 + C\|u\|_{\mathcal{H}_l^m}^4,
 \end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
 & \|D^{\tilde{\alpha}}(\Theta\phi') \cdot D^{\alpha-\tilde{\alpha}}\partial_y u\|_{L^2_{k+l-1}(\Omega)} \|D^{\alpha} u\|_{L^2_{k+l}(\Omega)} \\
 & \leq \|\partial_{\tau}^{\tilde{\beta}}\Theta\|_{L^{\infty}(\mathbb{T}_x)} \|D^{\alpha-\tilde{\alpha}}\partial_y u\|_{L^2_{k+l-1}(\Omega)} \|D^{\alpha} u\|_{L^2_{k+l}(\Omega)} \\
 & \leq C\|\partial_{\tau}^{\tilde{\beta}}\Theta\|_{H^1(\mathbb{T}_x)}^2 + C\|u\|_{\mathcal{H}_l^m}^4,
 \end{aligned} \tag{3.26}$$

provided $m \geq 4$. Then, substituting (3.24)-(3.26) into (3.23), we obtain directly

$$\begin{aligned} & \left| -2(k+l)\mu \int_{\Omega} D^{\alpha}[(\theta + \Theta\phi' + 1)\partial_y u] \cdot \langle y \rangle^{2k+2l-1} D^{\alpha} u dx dy \right| \\ & \leq \frac{\mu}{12} \|\partial_y D^{\alpha} u(t)\|_{L^2_{k+l}(\Omega)}^2 + C(\|(u, \theta)(t)\|_{\mathcal{H}_t^m}^4 + 1) + C \sum_{|\beta| \leq m+1} \|\partial_{\tau}^{\beta} \Theta(t)\|_{L^2(\mathbb{T}_x)}^4. \end{aligned} \quad (3.27)$$

Step 4: Finally, we deal with the term $-\mu \int_{\mathbb{T}_x} D^{\alpha}[(\theta + 1)\partial_y u] \cdot D^{\alpha} u|_{y=0} dx$ on the righthand side of (3.17).

Case 1: $|\alpha| \leq m-1$. Indeed, we can apply the simple trace estimate to get that

$$\begin{aligned} & \left| -\mu \int_{\mathbb{T}_x} D^{\alpha}[(\theta + 1)\partial_y u] \cdot D^{\alpha} u|_{y=0} dx \right| \\ & \leq \mu \|D^{\alpha}[(\theta + 1)\partial_y u]\|_{L^2(\Omega)} \|\partial_y D^{\alpha} u\|_{L^2(\Omega)} + \mu \|\partial_y D^{\alpha}[(\theta + 1)\partial_y u]\|_{L^2(\Omega)} \|D^{\alpha} u\|_{L^2(\Omega)}. \end{aligned} \quad (3.28)$$

By virtue of (A.3) and Cauchy inequality, one arrives at for $m \geq 4$ that

$$\|D^{\alpha}[(\theta + 1)\partial_y u]\|_{L^2(\Omega)} \|\partial_y D^{\alpha} u\|_{L^2(\Omega)} \leq C\|\theta\|_{\mathcal{H}_0^{m-1}} \|\partial_y u\|_{\mathcal{H}_0^{m-1}} + \|\partial_y D^{\alpha} u\|_{L^2(\Omega)}^2 \leq C\|(u, \theta)\|_{\mathcal{H}_0^m}^2, \quad (3.29)$$

and

$$\begin{aligned} & \|\partial_y D^{\alpha}[(\theta + 1)\partial_y u]\|_{L^2(\Omega)} \|D^{\alpha} u\|_{L^2(\Omega)} \\ & \leq \|D^{\alpha}(\partial_y \theta \partial_y u)\|_{L^2(\Omega)} \|D^{\alpha} u\|_{L^2(\Omega)} + \|D^{\alpha}[(\theta + 1)\partial_y^2 u]\|_{L^2(\Omega)} \|D^{\alpha} u\|_{L^2(\Omega)} \\ & \leq C\|\partial_y \theta\|_{\mathcal{H}_0^{m-1}} \|\partial_y u\|_{\mathcal{H}_0^{m-1}} \|D^{\alpha} u\|_{L^2(\Omega)} + C\|\theta\|_{\mathcal{H}_0^{m-1}} \|\partial_y^2 u\|_{\mathcal{H}_0^{m-1}} \|D^{\alpha} u\|_{L^2(\Omega)} \\ & \quad + \|D^{\alpha} \partial_y^2 u\|_{L^2(\Omega)} \|D^{\alpha} u\|_{L^2(\Omega)} \\ & \leq \delta_1 \|\partial_y u\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1}(\|(u, \theta)\|_{\mathcal{H}_0^m}^4 + 1). \end{aligned} \quad (3.30)$$

Substituting (3.29) and (3.30) into (3.28), we find

$$\left| -\mu \int_{\mathbb{T}_x} D^{\alpha}[(\theta + 1)\partial_y u] \cdot D^{\alpha} u|_{y=0} dx \right| \leq \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1}(\|(u, \theta)(t)\|_{\mathcal{H}_0^m}^4 + 1). \quad (3.31)$$

Case 2: $|\alpha| = m$. In view of $|\alpha| = |\beta| + k = m$, $|\beta| \leq m-1$, we deduce that $k \geq 1$. Let $\gamma \triangleq \alpha - E_3 = (\beta, k-1)$, then $|\gamma| \leq m-1$. Then, it is easy to deduce that

$$\begin{aligned} & -\mu \int_{\mathbb{T}_x} D^{\alpha}[(\theta + 1)\partial_y u] \cdot D^{\alpha} u|_{y=0} dx \\ & = -\mu \int_{\mathbb{T}_x} D^{\gamma}[\partial_y \theta \partial_y u + (\theta + 1)\partial_y^2 u] \cdot D^{\alpha} u|_{y=0} dx \\ & = -\mu \int_{\mathbb{T}_x} D^{\gamma}(\partial_y \theta \partial_y u) \cdot D^{\alpha} u|_{y=0} dx - \mu \int_{\mathbb{T}_x} (\theta + 1) D^{\gamma} \partial_y^2 u \cdot D^{\alpha} u|_{y=0} dx \\ & \quad - \mu \sum_{0 < \tilde{\gamma} \leq \gamma} \int_{\mathbb{T}_x} \begin{pmatrix} \gamma \\ \tilde{\gamma} \end{pmatrix} D^{\tilde{\gamma}} \theta D^{\tilde{\gamma}-\gamma} \partial_y^2 u \cdot D^{\alpha} u \Big|_{y=0} dx \\ & \triangleq G_1 + G_2 + G_3. \end{aligned} \quad (3.32)$$

Estimate for G_1 : We apply the inequality (A.3) and Cauchy inequality to deduce for $m \geq 4$

$$\begin{aligned} |G_1| & \leq \mu \|\partial_y D^{\gamma}(\partial_y \theta \partial_y u)\|_{L^2(\Omega)} \|D^{\alpha} u\|_{L^2(\Omega)} + \mu \|D^{\gamma}(\partial_y \theta \partial_y u)\|_{L^2(\Omega)} \|\partial_y D^{\alpha} u\|_{L^2(\Omega)} \\ & \leq \mu(\|\partial_y^2 \theta\|_{\mathcal{H}_0^{m-1}} \|\partial_y u\|_{\mathcal{H}_0^{m-1}} + \|\partial_y \theta\|_{\mathcal{H}_0^{m-1}} \|\partial_y^2 u\|_{\mathcal{H}_0^{m-1}}) \|D^{\alpha} u\|_{L^2(\Omega)} \\ & \quad + \mu \|\partial_y \theta\|_{\mathcal{H}_0^{m-1}} \|\partial_y u\|_{\mathcal{H}_0^{m-1}} \|\partial_y D^{\alpha} u\|_{L^2(\Omega)} \\ & \leq \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \kappa \|\partial_y \theta(t)\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1} \|(u, \theta)(t)\|_{\mathcal{H}_0^m}^4. \end{aligned} \quad (3.33)$$

Estimate for G_3 : With the help of (A.3) and Cauchy inequality, it is easy to check that for $m \geq 5$

$$\begin{aligned}
 & \left| \mu \int_{\mathbb{T}_x} D^{\tilde{\gamma}} \theta D^{\tilde{\gamma}-\gamma} \partial_y^2 u \cdot D^\alpha u \Big|_{y=0} dx \right| \\
 & \leq C\mu \|\partial_y(D^{\tilde{\gamma}} \theta D^{\gamma-\tilde{\gamma}} \partial_y^2 u)\|_{L^2(\Omega)} \|D^\alpha u\|_{L^2(\Omega)} + C\mu \|D^{\tilde{\gamma}} \theta D^{\gamma-\tilde{\gamma}} \partial_y^2 u\|_{L^2(\Omega)} \|\partial_y D^\alpha u\|_{L^2(\Omega)} \\
 & \leq C\mu (\|D^{\tilde{\gamma}-E_i} D^{E_i} \partial_y \theta D^{\gamma-\tilde{\gamma}} \partial_y^2 u\|_{L^2(\Omega)} + \|D^{\tilde{\gamma}-E_i} D^{E_i} \theta D^{\gamma-\tilde{\gamma}} \partial_y^3 u\|_{L^2(\Omega)}) \|D^\alpha u\|_{L^2(\Omega)} \\
 & \quad + C\mu \|D^{\tilde{\gamma}-E_i} D^{E_i} \theta D^{\gamma-\tilde{\gamma}} \partial_y^2 u\|_{L^2(\Omega)} \|\partial_y D^\alpha u\|_{L^2(\Omega)} \\
 & \leq C\mu (\|D^{E_i} \partial_y \theta\|_{\mathcal{H}_0^{m-2}} \|\partial_y^2 u\|_{\mathcal{H}_0^{m-2}} + \|D^{E_i} \theta\|_{\mathcal{H}_0^{m-2}} \|\partial_y^3 u\|_{\mathcal{H}_0^{m-2}}) \|D^\alpha u\|_{L^2(\Omega)} \\
 & \quad + C\mu \|D^{E_i} \theta\|_{\mathcal{H}_0^{m-2}} \|\partial_y^2 u\|_{\mathcal{H}_0^{m-2}} \|\partial_y D^\alpha u\|_{L^2(\Omega)} \\
 & \leq \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1} (\|(u, \theta)(t)\|_{\mathcal{H}_0^m}^4 + 1),
 \end{aligned}$$

which, implies that

$$|G_3| \leq \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1} (\|(u, \theta)(t)\|_{\mathcal{H}_0^m}^4 + 1). \quad (3.34)$$

Estimate for G_2 : Indeeet, by virtue of the equation (2.6)₁, it is easy to deduce that

$$\begin{aligned}
 (\theta + \Theta\phi' + 1)D^\gamma \partial_y^2 u &= D^\gamma \partial_t u + D^\gamma I_1 - \sum_{\tilde{\gamma} \leq \gamma} \begin{pmatrix} \gamma \\ \tilde{\gamma} \end{pmatrix} D^{\tilde{\gamma}} (\partial_y \theta + \Theta\phi'') D^{\gamma-\tilde{\gamma}} \partial_y u \\
 &\quad - \sum_{0 < \tilde{\gamma} \leq \gamma} \begin{pmatrix} \gamma \\ \tilde{\gamma} \end{pmatrix} D^{\tilde{\gamma}} (\theta + \Theta\phi') D^{\gamma-\tilde{\gamma}} \partial_y^2 u - D^\gamma r_1,
 \end{aligned}$$

which yields directly

$$\begin{aligned}
 G_2 &= \mu \int_{\mathbb{T}_x} D^\gamma \partial_t u D^\alpha u \Big|_{y=0} dx + \mu \int_{\mathbb{T}_x} D^\gamma I_1 D^\alpha u \Big|_{y=0} dx \\
 &\quad - \mu \sum_{\tilde{\gamma} \leq \gamma} \begin{pmatrix} \gamma \\ \tilde{\gamma} \end{pmatrix} \int_{\mathbb{T}_x} D^{\tilde{\gamma}} \partial_y \theta D^{\gamma-\tilde{\gamma}} \partial_y u D^\alpha u \Big|_{y=0} dx \\
 &\quad - \mu \sum_{0 < \tilde{\gamma} \leq \gamma} \begin{pmatrix} \gamma \\ \tilde{\gamma} \end{pmatrix} \int_{\mathbb{T}_x} D^{\tilde{\gamma}} \theta D^{\gamma-\tilde{\gamma}} \partial_y^2 u D^\alpha u \Big|_{y=0} dx \\
 &\quad - \mu \int_{\mathbb{T}_x} D^\gamma r_1 D^\alpha u \Big|_{y=0} dx \\
 &\triangleq G_{21} + G_{22} + G_{23} + G_{24} + G_{25}.
 \end{aligned} \quad (3.35)$$

In view of the Cauchy inequality, one arrives at

$$\begin{aligned}
 |G_{21}| &\leq \mu \|\partial_y D^{\gamma+e_1} u\|_{L^2(\Omega)} \|D^\alpha u\|_{L^2(\Omega)} + \mu \|D^{\gamma+e_1} u\|_{L^2(\Omega)} \|\partial_y D^\alpha u\|_{L^2(\Omega)} \\
 &\leq \frac{\mu}{12} \|\partial_y D^\alpha u(t)\|_{L^2(\Omega)}^2 + C\|u(t)\|_{\mathcal{H}_0^m}^2.
 \end{aligned} \quad (3.36)$$

By virtue of the definition of I_1 , it is easy to deduce that

$$\begin{aligned}
 G_{22} &= \mu \int_{\mathbb{T}_x} D^\gamma (uu_x) D^\alpha u \Big|_{y=0} dx + \mu \int_{\mathbb{T}_x} D^\gamma (vu_y) D^\alpha u \Big|_{y=0} dx \\
 &\quad - \mu \int_{\mathbb{T}_x} D^\gamma (hh_x) D^\alpha u \Big|_{y=0} dx - \mu \int_{\mathbb{T}_x} D^\gamma (gh_y) D^\alpha u \Big|_{y=0} dx \\
 &\triangleq G_{221} + G_{222} + G_{223} + G_{224}.
 \end{aligned} \quad (3.37)$$

One applies the inequality (A.3) and Cauchy inequality to obtain

$$\begin{aligned}
 |G_{221}| &\leq \mu \|\partial_y D^\gamma(uu_x)\|_{L^2(\Omega)} \|D^\alpha u\|_{L^2(\Omega)} + \mu \|D^\gamma(uu_x)\|_{L^2(\Omega)} \|\partial_y D^\alpha u\|_{L^2(\Omega)} \\
 &\leq \mu (\|\partial_y u\|_{\mathcal{H}_0^{m-1}} \|\partial_x u\|_{\mathcal{H}_0^{m-1}} + \|u\|_{\mathcal{H}_0^{m-1}} \|\partial_y \partial_x u\|_{\mathcal{H}_0^{m-1}}) \|D^\alpha u\|_{L^2(\Omega)} \\
 &\quad + \mu (\|u\|_{\mathcal{H}_0^{m-1}} \|\partial_x u\|_{\mathcal{H}_0^{m-1}} + \|u\|_{\mathcal{H}_0^{m-1}} \|\partial_x u\|_{\mathcal{H}_0^{m-1}}) \|\partial_y D^\alpha u\|_{L^2(\Omega)} \\
 &\leq \delta_1 \mu \|\partial_y u\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} (\|u\|_{\mathcal{H}_0^m}^4 + 1),
 \end{aligned} \tag{3.38}$$

provided $m \geq 3$. Following the idea as Liu et al.[42], it is easy to obtain the following estimate

$$|G_{222}| \leq \delta_1 \mu \|\partial_y u\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} (\|u\|_{\mathcal{H}_0^m}^4 + 1). \tag{3.39}$$

Similarly, it is easy to deduce that

$$\begin{aligned}
 |G_{223}| &\leq \delta_1 \mu \|\partial_y u\|_{\mathcal{H}_0^m}^2 + \delta_1 \nu \|\partial_y h\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} (\|(u, h)\|_{\mathcal{H}_0^m}^4 + 1), \\
 |G_{224}| &\leq \delta_1 \mu \|\partial_y u\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} (\|(u, h)\|_{\mathcal{H}_0^m}^4 + 1).
 \end{aligned} \tag{3.40}$$

Substituting the estimate (3.38)-(3.40) into (3.37), one obtains that

$$|G_{22}| \leq \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \nu \|\partial_y h(t)\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} (\|(u, h)(t)\|_{\mathcal{H}_0^m}^4 + 1). \tag{3.41}$$

In view of the trace inequality (A.1) and Sobolev inequality (A.3), we find for $m \geq 4$

$$\begin{aligned}
 &\mu \|D^{\tilde{\gamma}} \partial_y \theta \ D^{\gamma-\tilde{\gamma}} \partial_y u \ D^\alpha u\|_{y=0} \|_{L^1(\mathbb{T}_x)} \\
 &\leq \mu \|\partial_y (D^{\tilde{\gamma}} \partial_y \theta \ D^{\gamma-\tilde{\gamma}} \partial_y u)\|_{L^2(\Omega)} \|D^\alpha u\|_{L^2(\Omega)} + \mu \|D^{\tilde{\gamma}} \partial_y \theta \ D^{\gamma-\tilde{\gamma}} \partial_y u\|_{L^2(\Omega)} \|\partial_y D^\alpha u\|_{L^2(\Omega)} \\
 &\leq \mu \|\partial_y^2 \theta\|_{\mathcal{H}_0^{m-1}} \|\partial_y u\|_{\mathcal{H}_0^{m-1}} \|D^\alpha u\|_{L^2(\Omega)} + \mu \|\partial_y \theta\|_{\mathcal{H}_0^{m-1}} \|\partial_y^2 u\|_{\mathcal{H}_0^{m-1}} \|D^\alpha u\|_{L^2(\Omega)} \\
 &\quad + \mu \|\partial_y \theta\|_{\mathcal{H}_0^{m-1}} \|\partial_y u\|_{\mathcal{H}_0^{m-1}} \|\partial_y D^\alpha u\|_{L^2(\Omega)} \\
 &\leq \delta_1 \mu \|\partial_y u\|_{\mathcal{H}_0^m}^2 + \delta_1 \kappa \|\partial_y \theta\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} (\|(u, \theta)\|_{\mathcal{H}_0^m}^4 + 1),
 \end{aligned}$$

which, implies that

$$|G_{23}| \leq \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \kappa \|\partial_y \theta(t)\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} (\|(u, \theta)(t)\|_{\mathcal{H}_0^m}^4 + 1). \tag{3.42}$$

Similarly, it is easy to deduce that

$$|G_{24}| \leq \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} (\|(u, \theta)(t)\|_{\mathcal{H}_0^m}^4 + 1). \tag{3.43}$$

In view of the definition of r_1 (see (2.9)) and trace inequality (A.1), one attains directly

$$\begin{aligned}
 |G_{25}| &= \left| \mu \int_{\mathbb{T}_x} D^\gamma P_x \cdot D^\alpha u dx \right| \\
 &\leq \mu \|D^\gamma P_x\|_{L^2(\mathbb{T}_x)} \|D^\alpha u\|_{L^2(\mathbb{T}_x)} \\
 &\leq \sqrt{2} \mu \|D^\gamma P_x\|_{L^2(\mathbb{T}_x)} \|D^\alpha u\|_{L^2(\Omega)}^{\frac{1}{2}} \|D^\alpha \partial_y u\|_{L^2(\Omega)}^{\frac{1}{2}} \\
 &\leq \frac{\mu}{12} \|\partial_y D^\alpha u\|_{L^2(\Omega)}^2 + C (\|u\|_{\mathcal{H}_0^m}^2 + \|D^\gamma P_x\|_{L^2(\mathbb{T}_x)}^2).
 \end{aligned} \tag{3.44}$$

Substituting (3.36), (3.41), (3.42), (3.43) and (3.44) into (3.35), we find

$$\begin{aligned}
 |G_2| &\leq \frac{\mu}{6} \|\partial_y D^\alpha u(t)\|_{L^2(\Omega)}^2 + \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \kappa \|\partial_y \theta(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \nu \|\partial_y h(t)\|_{\mathcal{H}_0^m}^2 \\
 &\quad + C \delta_1^{-1} (\|(u, \theta, h)(t)\|_{\mathcal{H}_0^m}^4 + 1) + C \sum_{|\beta| \leq m-1} \|\partial_\tau^\beta P_x(t)\|_{L^2(\mathbb{T}_x)}^2,
 \end{aligned}$$

which, together with (3.33) and (3.34), gives for $|\alpha| = m$ that

$$\begin{aligned} & |\mu \int_{\mathbb{T}_x} D^\alpha [(\theta + 1) \partial_y u] \cdot D^\alpha u|_{y=0} dx dy| \\ & \leq \frac{\mu}{6} \|\partial_y D^\alpha u(t)\|_{L^2(\Omega)}^2 + \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \kappa \|\partial_y \theta(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \nu \|\partial_y h(t)\|_{\mathcal{H}_0^m}^2 \\ & \quad + C \delta_1^{-1} (\|(u, \theta, h)(t)\|_{\mathcal{H}_0^m}^4 + 1) + C \sum_{|\beta| \leq m-1} \|\partial_\tau^\beta P_x(t)\|_{L^2(\mathbb{T}_x)}^2. \end{aligned} \quad (3.45)$$

The combination of (3.31) and (3.45) yields for $|\alpha| \leq m$ that

$$\begin{aligned} & |\mu \int_{\mathbb{T}_x} D^\alpha [(\theta + 1) \partial_y u] \cdot D^\alpha u|_{y=0} dx| \\ & \leq \frac{\mu}{6} \|\partial_y D^\alpha u(t)\|_{L^2(\Omega)}^2 + \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \kappa \|\partial_y \theta(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \nu \|\partial_y h(t)\|_{\mathcal{H}_0^m}^2 \\ & \quad + C \delta_1^{-1} (\|(u, \theta, h)(t)\|_{\mathcal{H}_0^m}^4 + 1) + C \sum_{|\beta| \leq m-1} \|\partial_\tau^\beta P_x(t)\|_{L^2(\mathbb{T}_x)}^2. \end{aligned} \quad (3.46)$$

Substituting (3.18), (3.22), (3.27) and (3.46) into (3.17), one finds that

$$\begin{aligned} & \mu \int_{\Omega} \partial_y D^\alpha [(\theta + \Theta \phi' + 1) \partial_y u] \cdot \langle y \rangle^{2k+2l} D^\alpha u dx dy \\ & \leq -\frac{\mu}{2} \|D^\alpha \partial_y u(t)\|_{L_{k+l}^2(\Omega)}^2 + \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \kappa \|\partial_y \theta(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \nu \|\partial_y h(t)\|_{\mathcal{H}_0^m}^2 \\ & \quad + C \delta_1^{-1} (\|(u, \theta, h)(t)\|_{\mathcal{H}_0^m}^4 + 1) + \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta \Theta(t)\|_{L^2(\mathbb{T}_x)}^4 + \sum_{|\beta| \leq m-1} \|\partial_\tau^\beta P_x(t)\|_{L^2(\mathbb{T}_x)}^2. \end{aligned} \quad (3.47)$$

Similarly (or following the idea as Liu et al.[42]), one finds that

$$\begin{aligned} & \nu \int_{\Omega} \partial_y^2 D^\alpha h \cdot \langle y \rangle^{2k+2l} D^\alpha h dx dy \\ & \leq -\frac{\nu}{2} \|D^\alpha \partial_y h(t)\|_{L_{k+l}^2(\Omega)}^2 + \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \nu \|\partial_y h(t)\|_{\mathcal{H}_0^m}^2 \\ & \quad + C \delta_1^{-1} (\|(u, h)(t)\|_{\mathcal{H}_0^m}^4 + 1), \end{aligned} \quad (3.48)$$

and

$$\begin{aligned} & \kappa \int_{\Omega} \partial_y^2 D^\alpha \theta \cdot \langle y \rangle^{2k+2l} D^\alpha \theta dx dy \\ & \leq -\frac{\kappa}{2} \|D^\alpha \partial_y \theta(t)\|_{L_{k+l}^2(\Omega)}^2 + \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \delta_1 \kappa \|\partial_y \theta(t)\|_{\mathcal{H}_0^m}^2 \\ & \quad + \delta_1 \nu \|\partial_y h(t)\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} (\|(u, \theta)(t)\|_{\mathcal{H}_0^m}^6 + 1). \end{aligned} \quad (3.49)$$

Therefore, the combination of (3.47)-(3.49) completes the proof of (3.14). \square

Proof of (3.15). We will first handle the term $-\int_{\Omega} D^\alpha I_2 \cdot \langle y \rangle^{2k+2l} D^\alpha \theta dx dy$. As we know that

$$\begin{aligned} I_2 \triangleq & c_v [(u + U \phi') \partial_x + (v - U_x \phi) \partial_y] \theta + c_v \Theta_x \phi' u + c_v \Theta \phi'' v - \mu \theta (u_y)^2 - \mu (U \phi'')^2 \theta \\ & - 2\mu U \phi'' \theta u_y - \mu \Theta \phi' (u_y)^2 - 2\mu \Theta U \phi' \phi'' u_y - 2\mu U \phi'' u_y - \mu (u_y)^2 - \nu (h_y)^2 - 2\nu H \phi' h_y. \end{aligned}$$

Then it is easy to deduce that

$$D^\alpha I_2 = I_{21} + I_{22} + I_{23}, \quad (3.50)$$

where

$$\begin{aligned}
 I_{21} &\triangleq c_v[(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]D^\alpha\theta, \\
 I_{22} &\triangleq c_v[D^\alpha, (u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\theta, \\
 I_{23} &\triangleq D^\alpha(c_v\Theta_x\phi'u + c_v\Theta\phi''v - \mu\theta(u_y)^2 - \mu(U\phi'')^2\theta - 2\mu U\phi''\theta u_y - \mu\Theta\phi'(u_y)^2) \\
 &\quad - D^\alpha(2\mu\Theta U\phi'\phi''u_y + 2\mu U\phi''u_y + \mu(u_y)^2 + \nu(h_y)^2 + 2\nu H\phi'h_y).
 \end{aligned} \tag{3.51}$$

Step 1: Integrating by part and applying the divergence free condition of velocity, we find for $m \geq 3$

$$\begin{aligned}
 & - \int_{\Omega} I_{21} \cdot \langle y \rangle^{2k+2l} D^\alpha \theta dx dy \\
 &= (2k + 2l) \int_{\Omega} \langle y \rangle^{2k+2l-1} (v - U_x\phi) |D^\alpha \theta|^2 dx dy \\
 &\leq C \|\langle y \rangle^{-1} v\|_{L^\infty(\Omega)} \|D^\alpha \theta\|_{L^2_{k+l}(\Omega)} + C \|\langle y \rangle^{-1} U_x\phi\|_{L^\infty(\Omega)} \|D^\alpha \theta\|_{L^2_{k+l}(\Omega)} \\
 &\leq C \|u_x\|_{L^\infty(\Omega)} \|D^\alpha \theta\|_{L^2_{k+l}(\Omega)} + C \|U_x\|_{L^\infty(\mathbb{T}_x)} \|D^\alpha \theta\|_{L^2_{k+l}(\Omega)} \\
 &\leq C \|u_x\|_{H^2(\Omega)} \|\theta\|_{\mathcal{H}_l^m}^2 + C \|U_x\|_{H^1(\mathbb{T}_x)} \|\theta\|_{\mathcal{H}_l^m}^2 \\
 &\leq C(\|(u, \theta)\|_{\mathcal{H}_l^m}^4 + 1) + C \|U_x\|_{H^1(\mathbb{T}_x)}^2,
 \end{aligned} \tag{3.52}$$

where we have used the Hardy type inequality (A.5).

Step 2: It is easy to deduce that

$$\| [D^\alpha, u\partial_x + v\partial_y]\theta \|_{L^2_{k+l}(\Omega)} \lesssim \sum_{0 < \tilde{\alpha} \leq \alpha} \|D^\alpha u \cdot D^{\alpha-\tilde{\alpha}}\theta_x\|_{L^2_{k+l}(\Omega)} + \sum_{0 < \tilde{\alpha} \leq \alpha} \|D^\alpha v \cdot D^{\alpha-\tilde{\alpha}}\theta_y\|_{L^2_{k+l}(\Omega)}. \tag{3.53}$$

Case 1: $\tilde{k} = 0$. One can infer that $D^{\tilde{\alpha}} = \partial_\tau^{\tilde{\beta}}$, and $\tilde{\beta} \geq e_i, i = 1$ or $2, |k| \leq m - 1$. Then, we find

$$\|D^{\tilde{\alpha}} u D^{\alpha-\tilde{\alpha}}\theta_x\|_{L^2_{k+l}(\Omega)} = \|\partial_\tau^{\tilde{\beta}-e_i} \partial_\tau^{e_i} u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^k \theta_x\|_{L^2_{k+l}(\Omega)} \leq \|\partial_\tau^{e_i} u\|_{\mathcal{H}_0^{m-1}} \|\theta_x\|_{\mathcal{H}_l^{m-1}} \leq \|u\|_{\mathcal{H}_0^m} \|\theta\|_{\mathcal{H}_l^m}, \tag{3.54}$$

provided that $m \geq 4$. On the other hand, it is easy to deduce that

$$\|D^{\tilde{\alpha}} v \cdot D^{\alpha-\tilde{\alpha}}\theta_y\|_{L^2_{k+l}(\Omega)} = \|\partial_\tau^{\tilde{\beta}} \partial_y^{-1} \partial_x u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^k \theta_y\|_{L^2_{k+l}(\Omega)}. \tag{3.55}$$

If $|\alpha| = |\beta| + k \leq m - 1$, one applies the inequality (A.3) to obtain that

$$\|\partial_\tau^{\tilde{\beta}} \partial_y^{-1} \partial_x u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^k \theta_y\|_{L^2_{k+l}(\Omega)} \leq \|u_x\|_{\mathcal{H}_0^{m-1}} \|\theta_y\|_{\mathcal{H}_{l+1}^{m-1}} \leq \|u\|_{\mathcal{H}_0^m} \|\theta\|_{\mathcal{H}_l^m}, \tag{3.56}$$

provided that $m \geq 4$. If $|\alpha| = |\beta| + k = m$, then it infers that $k \geq 1$ and one finds

$$\begin{aligned}
 & \|\partial_\tau^{\tilde{\beta}} \partial_y^{-1} \partial_x u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^k \theta_y\|_{L^2_{k+l}(\Omega)} = \|\partial_\tau^{\tilde{\beta}} \partial_y^{-1} \partial_x u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-1} \theta_{yy}\|_{L^2_{k+l}(\Omega)} \\
 & \leq \|\partial_x \partial_\tau^{e_i} u\|_{\mathcal{H}_0^{m-2}} \|\partial_y^2 \theta\|_{\mathcal{H}_{l+2}^{m-2}} \leq \|u\|_{\mathcal{H}_0^m} \|\theta\|_{\mathcal{H}_l^m},
 \end{aligned} \tag{3.57}$$

provided that $m \geq 5$. The combination of (3.54)-(3.57) yields directly that

$$\| [D^\alpha, u\partial_x + v\partial_y]\theta \|_{L^2_{k+l}(\Omega)} \leq C \|(u, \theta)(t)\|_{\mathcal{H}_l^m}^2. \tag{3.58}$$

Case 2: $\tilde{k} \geq 1$. Then, we get that $\tilde{\alpha} \geq E_3$ and obtain

$$\|D^{\tilde{\alpha}} u \cdot D^{\alpha-\tilde{\alpha}}\theta_x\|_{L^2_{k+l}(\Omega)} = \|D^{\tilde{\alpha}-E_3} D^{E_3} u \cdot D^{\alpha-\tilde{\alpha}}\theta_x\|_{L^2_{k+l}(\Omega)} \leq \|\partial_x \theta\|_{\mathcal{H}_0^m} \|D^{E_3} u\|_{\mathcal{H}_{l+1}^{m-1}} \leq C \|(u, \theta)\|_{\mathcal{H}_l^m}^2. \tag{3.59}$$

On the other hand, one applies the inequality (A.3) and divergence free of velocity to deduce for $m \geq 4$

$$\|D^{\tilde{\alpha}} v D^{\alpha-\tilde{\alpha}} \theta_y\|_{L^2_{k+l}(\Omega)} = \|D^{\tilde{\alpha}-E_3} u_x D^{\alpha-\tilde{\alpha}} \theta_y\|_{L^2_{k+l}(\Omega)} \leq \|\partial_x u\|_{\mathcal{H}_0^{m-1}} \|\partial_y \theta\|_{\mathcal{H}_{l+1}^{m-1}} \leq C \|(u, \theta)\|_{\mathcal{H}_l^m}^2,$$

which, together with (3.59), yields that

$$\|[D^\alpha, u\partial_x + v\partial_y]\theta\|_{L^2_{k+l}(\Omega)} \leq C \|(u, \theta)\|_{\mathcal{H}_l^m}^2. \quad (3.60)$$

Substituting the combination of (3.58) and (3.60) into (3.53), we find for all $\tilde{k} \geq 0$

$$\|[D^\alpha, u\partial_x + v\partial_y]\theta\|_{L^2_{k+l}(\Omega)} \leq C \|(u, \theta)\|_{\mathcal{H}_l^m}^2. \quad (3.61)$$

On the other hand, it is easy to deduce that

$$\|[D^\alpha, U\phi'\partial_x - U_x\phi\partial_y]\theta\|_{L^2_{k+l}(\Omega)} \lesssim \sum_{0 < \tilde{\alpha} \leq \alpha} \|D^{\tilde{\alpha}}(U\phi') D^{\alpha-\tilde{\alpha}} \theta_x\|_{L^2_{k+l}(\Omega)} + \sum_{0 < \tilde{\alpha} \leq \alpha} \|D^{\tilde{\alpha}}(U_x\phi) D^{\alpha-\tilde{\alpha}} \theta_y\|_{L^2_{k+l}(\Omega)}. \quad (3.62)$$

In view of the fact $|\alpha - \tilde{\alpha}| \leq m - 1$, then one arrives at

$$\|D^{\tilde{\alpha}}(U\phi') D^{\alpha-\tilde{\alpha}} \theta_x\|_{L^2_{k+l}(\Omega)} \leq \|\langle y \rangle^{\tilde{k}} D^{\tilde{\alpha}}(U\phi')\|_{L^\infty(\Omega)} \|\langle y \rangle^{k+l-\tilde{k}} D^{\alpha-\tilde{\alpha}} \theta_x\|_{L^2(\Omega)} \leq \|\partial_\tau^{\tilde{\beta}} U\|_{L^\infty(\mathbb{T}_x)} \|\theta\|_{\mathcal{H}_l^m}, \quad (3.63)$$

and

$$\|D^{\tilde{\alpha}}(U_x\phi) D^{\alpha-\tilde{\alpha}} \theta_y\|_{L^2_{k+l}(\Omega)} \leq \|\partial_\tau^{\tilde{\beta}} U_x\|_{L^\infty(\mathbb{T}_x)} \|\theta\|_{\mathcal{H}_l^m} \quad (3.64)$$

Substituting (3.63) and (3.64) into (3.62), we find directly

$$\|[D^\alpha, U\phi'\partial_x - U_x\phi\partial_y]\theta\|_{L^2_{k+l}(\Omega)} \leq C \|\theta\|_{\mathcal{H}_l^m}^2 + \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta U\|_{L^2(\mathbb{T}_x)}^2,$$

which, together with (3.61), yields directly

$$-\int_{\Omega} I_{22} \cdot \langle y \rangle^{2k+2l} D^\alpha \theta dx dy \leq C(\|(u, \theta)(t)\|_{\mathcal{H}_l^m}^4 + 1) + \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta U(t)\|_{L^2(\mathbb{T}_x)}^4. \quad (3.65)$$

Step 3: Finally, we will give the estimate for $\|I_{23}\|_{L^2_{k+l}(\Omega)}$. Recall the definition of I_{23} (see (3.51))

$$\begin{aligned} I_{23} \triangleq & D^\alpha(c_v \Theta_x \phi' u + c_v \Theta \phi'' v - \mu \theta (u_y)^2 - \mu (U \phi'')^2 \theta - 2\mu U \phi'' \theta u_y - \mu \Theta \phi' (u_y)^2) \\ & - D^\alpha(2\mu \Theta U \phi' \phi'' u_y + 2\mu U \phi'' u_y + \mu (u_y)^2 + \nu (h_y)^2 + 2\nu H \phi' h_y). \end{aligned}$$

Estimate for $\|D^\alpha(\Theta_x \phi' u)\|_{L^2_{l+k}(\Omega)}$. In view of the definition of ϕ (see (2.2)), we can obtain

$$\begin{aligned} \|D^\alpha(\Theta_x \phi' u)\|_{L^2_{l+k}(\Omega)} & \leq C \sum_{\tilde{\alpha} \leq \alpha} \|D^{\tilde{\alpha}} u \cdot D^{\alpha-\tilde{\alpha}}(\Theta_x \phi')\|_{L^2_{l+k}(\Omega)} \\ & \leq C \sum_{\tilde{\alpha} \leq \alpha} \|\langle y \rangle^{l+\tilde{k}} D^{\tilde{\alpha}} u\|_{L^2(\Omega)} \|\langle y \rangle^{k-\tilde{k}} D^{\alpha-\tilde{\alpha}}(\Theta_x \phi')\|_{L^\infty(\Omega)} \\ & \leq C \|u\|_{\mathcal{H}_l^m} \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta \Theta\|_{L^2(\mathbb{T}_x)}. \end{aligned} \quad (3.66)$$

Estimate for $\|D^\alpha(\Theta \phi'' v)\|_{L^2_{l+k}(\Omega)}$. By virtue of the divergence free condition of velocity, one arrives at

$$\|D^\alpha(\Theta \phi'' v)\|_{L^2_{l+k}(\Omega)} \leq \sum_{\tilde{\alpha} \leq \alpha} \|D^{\tilde{\alpha}} v D^{\alpha-\tilde{\alpha}}(\Theta \phi'')\|_{L^2_{k+l}(\Omega)} \leq \sum_{\tilde{\alpha} \leq \alpha} \|D^{\tilde{\alpha}+E_2} \partial_y^{-1} u D^{\alpha-\tilde{\alpha}}(\Theta \phi'')\|_{L^2_{k+l}(\Omega)}. \quad (3.67)$$

If $\tilde{k} = 0$, the application of Hardy type inequality (A.7) yields directly

$$\begin{aligned}
& \|D^{\tilde{\alpha}+e_2}\partial_y^{-1}u \cdot D^{\alpha-\tilde{\alpha}}(\Theta\phi'')\|_{L_{k+l}^2(\Omega)} \\
& \leq \| \langle y \rangle^{-1} \partial_\tau^{\tilde{\beta}+e_2} \partial_y^{-1}u \|_{L^2(\Omega)} \| \langle y \rangle^{k+l+1} \partial_\tau^{\beta-\tilde{\beta}} \partial_y^k(\Theta\phi'') \|_{L^\infty(\Omega)} \\
& \leq \| \partial_\tau^{\tilde{\beta}+e_2} u \|_{L^2(\Omega)} \| \partial_\tau^{\beta-\tilde{\beta}} \Theta \|_{L^\infty(\mathbb{T}_x)} \\
& \leq C \|u\|_{\mathcal{H}_0^m} \| \partial_\tau^{\beta-\tilde{\beta}} \Theta \|_{L^\infty(\mathbb{T}_x)}.
\end{aligned} \tag{3.68}$$

If $\tilde{k} \geq 1$, it is easy to deduce that

$$\begin{aligned}
& \|D^{\tilde{\alpha}+e_2}\partial_y^{-1}u \cdot D^{\alpha-\tilde{\alpha}}(\Theta\phi'')\|_{L_{k+l}^2(\Omega)} \\
& \leq \| \partial_\tau^{\tilde{\beta}+e_2} \partial_y^{\tilde{k}-1}u \cdot \partial_\tau^{\beta-\tilde{\beta}}(\Theta\phi'') \|_{L_{k+l}^2(\Omega)} \\
& \leq \| \langle y \rangle^{\tilde{k}-1} \partial_\tau^{\tilde{\beta}+e_2} \partial_y^{\tilde{k}-1}u \|_{L^2(\Omega)} \| \langle y \rangle^{l+k-\tilde{k}+1} \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-\tilde{k}}(\Theta\phi'') \|_{L^\infty(\Omega)} \\
& \leq C \|u\|_{\mathcal{H}_0^m} \| \partial_\tau^{\beta-\tilde{\beta}} \Theta \|_{L^\infty(\mathbb{T}_x)}.
\end{aligned} \tag{3.69}$$

Then, substituting the estimates (3.68) and (3.69) into (3.67), we find

$$\|D^\alpha(\Theta\phi''v)\|_{L_{l+k}^2(\Omega)} \leq C \|u\|_{\mathcal{H}_0^m} \sum_{|\beta| \leq m+1} \| \partial_\tau^\beta \Theta \|_{L^2(\mathbb{T}_x)}. \tag{3.70}$$

Estimate for $\|D^\alpha(\theta(u_y)^2)\|_{L_{l+k}^2(\Omega)}$. Indeed, it is easy to check that

$$\begin{aligned}
& \|D^\alpha(\theta(u_y)^2)\|_{L_{l+k}^2(\Omega)} \\
& \leq \| \theta \partial_y u \cdot \partial_y D^\alpha u \|_{L_{l+k}^2(\Omega)} + C \sum_{0 < \tilde{\alpha} \leq \alpha} \| \theta D^{\tilde{\alpha}-E_i} D^{E_i} \partial_y u \cdot D^{\alpha-\tilde{\alpha}} \partial_y u \|_{L_{l+k}^2(\Omega)} \\
& \quad + C \sum_{0 < \alpha \leq \alpha} \| D^{\tilde{\alpha}-E_i} D^{E_i} \theta \cdot D^{\alpha-\tilde{\alpha}} ((\partial_y u)^2) \|_{L_{l+k}^2(\Omega)}.
\end{aligned} \tag{3.71}$$

On one hand, we apply the Sobolev inequality to deduce that

$$\begin{aligned}
& \| \theta \partial_y u D^\alpha \partial_y u \|_{L_{l+k}^2(\Omega)} \\
& \leq \| D^\alpha \partial_y u \|_{L_{l+k}^2(\Omega)} \| \theta \|_{L^\infty(\Omega)} \| \partial_y u \|_{L^\infty(\Omega)} \\
& \leq \| D^\alpha \partial_y u \|_{L_{l+k}^2(\Omega)} \| \theta \|_{H^2(\Omega)} \| \partial_y u \|_{H^2(\Omega)}.
\end{aligned} \tag{3.72}$$

On the other hand, the application of inequality (A.3) and Sobolev inequality yields directly

$$\begin{aligned}
& \| \theta D^{\tilde{\alpha}-E_i} D^{E_i} \partial_y u \cdot D^{\alpha-\tilde{\alpha}} \partial_y u \|_{L_{l+k}^2(\Omega)} \\
& \leq C \| \theta \|_{L^\infty} \| D^{E_i} \partial_y u \|_{\mathcal{H}_0^{m-1}} \| \partial_y u \|_{\mathcal{H}_{l+1}^{m-1}} \\
& \leq C \| \partial_y u \|_{\mathcal{H}_0^m} \| \theta \|_{H^2} \| u \|_{\mathcal{H}_l^m},
\end{aligned} \tag{3.73}$$

and

$$\begin{aligned}
& \| D^{\tilde{\alpha}-E_i} D^{E_i} \theta \cdot D^{\alpha-\tilde{\alpha}} ((\partial_y u)^2) \|_{L_{l+k}^2(\Omega)} \\
& \leq C \| D^{E_i} \theta \|_{\mathcal{H}_0^{m-1}} \| (\partial_y u)^2 \|_{\mathcal{H}_{l+1}^{m-1}} \leq C \| \theta \|_{\mathcal{H}_0^m} \| u \|_{\mathcal{H}_l^m}^2,
\end{aligned} \tag{3.74}$$

provided that $m \geq 4$. Substituting the estimates (3.72)-(3.74) into (3.71), we obtain for $m \geq 4$

$$\|D^\alpha(\theta(\partial_y u)^2)\|_{L_{l+k}^2(\Omega)} \leq \|D^\alpha \partial_y u\|_{L_{l+k}^2(\Omega)} \| (u, \theta) \|_{\mathcal{H}_l^m}^2 + \| \partial_y u \|_{\mathcal{H}_0^m} \| (u, \theta) \|_{\mathcal{H}_l^m}^2 + \| (u, \theta) \|_{\mathcal{H}_l^m}^3. \tag{3.75}$$

Estimate for $\|D^\alpha((U\phi'')^2\theta)\|_{L^2_{l+k}(\Omega)}$. By virtue of the definition of (2.2), it is easy to deduce that

$$\begin{aligned}
 \|D^\alpha((U\phi'')^2\theta)\|_{L^2_{l+k}(\Omega)} &\leq C \sum_{\tilde{\alpha} \leq \alpha} \|D^{\tilde{\alpha}}\theta D^{\alpha-\tilde{\alpha}}[(U)^2(\phi'')^2]\|_{L^2_{l+k}(\Omega)} \\
 &\leq C \sum_{\tilde{\alpha} \leq \alpha} \|\langle y \rangle^{\tilde{k}} D^{\tilde{\alpha}}\theta\|_{L^2(\Omega)} \|\langle y \rangle^{k-\tilde{k}+l} D^{\alpha-\tilde{\alpha}}[(U)^2(\phi'')^2]\|_{L^\infty(\Omega)} \\
 &\leq C \|\theta\|_{\mathcal{H}_0^m} \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta U\|_{L^2(\mathbb{T}_x)}^2.
 \end{aligned} \tag{3.76}$$

Similarly, we can find that

$$\begin{aligned}
 \|D^\alpha(\Theta U \phi' \phi'' u_y)\|_{L^2_{l+k}(\Omega)} &\leq C \|\partial_y u\|_{\mathcal{H}_0^m} \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(U, \Theta)\|_{L^2(\mathbb{T}_x)}^2, \\
 \|D^\alpha(U \phi'' u_y)\|_{L^2_{l+k}(\Omega)} &\leq C \|\partial_y u\|_{\mathcal{H}_0^m} \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta U\|_{L^2(\mathbb{T}_x)}, \\
 \|D^\alpha(U \phi'' \theta \partial_y u)\|_{L^2_{l+k}(\Omega)} &\leq C \|\partial_y u\|_{\mathcal{H}_0^m} \|\theta\|_{\mathcal{H}_0^m} \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta U\|_{L^2(\mathbb{T}_x)}.
 \end{aligned} \tag{3.77}$$

Estimate for $\|D^\alpha(H\phi' h_y)\|_{L^2_{l+k}(\Omega)}$. We apply the inequality (A.3) to deduce that

$$\begin{aligned}
 &\|D^\alpha(H\phi' h_y)\|_{L^2_{l+k}(\Omega)} \\
 &\leq C \sum_{0 < \tilde{\alpha} \leq \alpha} \|\langle y \rangle^{\tilde{k}} D^{\tilde{\alpha}}(H\phi')\|_{L^\infty(\Omega)} \|\langle y \rangle^{k-\tilde{k}+l} D^{\alpha-\tilde{\alpha}} h_y\|_{L^2(\Omega)} \\
 &\quad + \|H\phi'\|_{L^\infty(\Omega)} \|D^\alpha h_y\|_{L^2_{l+k}(\Omega)} \\
 &\leq \|H\|_{L^\infty(\mathbb{T}_x)} \|D^\alpha h_y\|_{L^2_{l+k}(\Omega)} + \|h\|_{\mathcal{H}_l^m} \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta H\|_{L^2(\mathbb{T}_x)}.
 \end{aligned} \tag{3.78}$$

Estimate for $\|D^\alpha[(\partial_y u)^2]\|_{L^2_{l+k}(\Omega)}$. By virtue of the inequality (A.3) and the Sobolev inequality, we find

$$\begin{aligned}
 &\|D^\alpha[(\partial_y u)^2]\|_{L^2_{l+k}(\Omega)} \\
 &\leq \|\partial_y u\|_{L^\infty(\Omega)} \|D^\alpha \partial_y u\|_{L^2_{l+k}(\Omega)} + C \sum_{0 < \tilde{\alpha} \leq \alpha} \|D^{\tilde{\alpha}-E_i} D^{E_i} \partial_y u\|_{L^2_{l+k}(\Omega)} \|D^{\alpha-\tilde{\alpha}} \partial_y u\|_{L^2_{l+k}(\Omega)} \\
 &\leq \|\partial_y u\|_{L^\infty(\Omega)} \|D^\alpha \partial_y u\|_{L^2_{l+k}(\Omega)} + C \|D^{E_i} \partial_y u\|_{\mathcal{H}_0^{m-1}} \|\partial_y u\|_{\mathcal{H}_{l+1}^{m-1}} \\
 &\leq \|\partial_y u\|_{H^2(\Omega)} \|D^\alpha \partial_y u\|_{L^2_{l+k}(\Omega)} + C \|\partial_y u\|_{\mathcal{H}_0^m} \|u\|_{\mathcal{H}_l^m},
 \end{aligned} \tag{3.79}$$

provided that $m \geq 4$. Similarly, it is easy to deduce that

$$\|D^\alpha[(\partial_y h)^2]\|_{L^2_{l+k}(\Omega)} \leq C \|\partial_y h\|_{H^2(\Omega)} \|D^\alpha \partial_y h\|_{L^2_{l+k}(\Omega)} + C \|\partial_y h\|_{\mathcal{H}_0^m} \|h\|_{\mathcal{H}_l^m}. \tag{3.80}$$

Estimate for $\|D^\alpha(\Theta \phi'(u_y)^2)\|_{L^2_{l+k}(\Omega)}$. In view of the inequality (A.3), it is easy to deduce

$$\begin{aligned}
 &\|D^\alpha(\Theta \phi'(u_y)^2)\|_{L^2_{l+k}(\Omega)} \\
 &\leq C \sum_{0 \leq \tilde{\alpha} < \alpha} \|\langle y \rangle^{\tilde{k}+l} D^{\tilde{\alpha}}(u_y^2)\|_{L^2(\Omega)} \|\langle y \rangle^{k-\tilde{k}} D^{\alpha-\tilde{\alpha}}(\Theta \phi')\|_{L^\infty(\Omega)} \\
 &\quad + \|\Theta \phi'\|_{L^\infty(\Omega)} \|\partial^\alpha(u_y^2)\|_{L^2_{k+l}(\Omega)} \\
 &\leq C \|\Theta\|_{H^1(\mathbb{T}_x)} (\|\partial_y u\|_{H^2(\Omega)} \|D^\alpha \partial_y u\|_{L^2_{l+k}(\Omega)} + \|\partial_y u\|_{\mathcal{H}_0^m} \|u\|_{\mathcal{H}_l^m}) \\
 &\quad + C \|\partial_y u\|_{\mathcal{H}_0^m} \|u\|_{\mathcal{H}_l^m} \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta \Theta\|_{L^2(\mathbb{T}_x)}.
 \end{aligned} \tag{3.81}$$

Combining the estimates (3.66), (3.70), (3.75)-(3.81) and applying the Cauchy inequality, we find

$$\begin{aligned} & - \int_{\Omega} D^{\alpha} I_{23} \cdot \langle y \rangle^{2k+2l} D^{\alpha} \theta dx dy \\ & \leq \frac{\mu}{4} \|D^{\alpha} \partial_y u(t)\|_{L^2_{l+k}(\Omega)} + \frac{\nu}{4} \|D^{\alpha} \partial_y h(t)\|_{L^2_{l+k}(\Omega)} + \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 \\ & \quad + \delta_1 \nu \|\partial_y h(t)\|_{\mathcal{H}_0^m}^2 + \delta_1^{-1} (\|(u, \theta, h)(t)\|_{\mathcal{H}_l^m}^8 + 1) + \sum_{|\beta| \leq m+2} \|\partial_{\tau}^{\beta}(U, \Theta, H)(t)\|_{L^2(\mathbb{T}_x)}^8, \end{aligned}$$

which, together with the estimates (3.52) and (3.65), yields

$$\begin{aligned} & - \int_{\Omega} D^{\alpha} I_2 \cdot \langle y \rangle^{2k+2l} D^{\alpha} \theta dx dy \\ & \leq \frac{\mu}{4} \|D^{\alpha} \partial_y u(t)\|_{L^2_{l+k}(\Omega)} + \frac{\nu}{4} \|D^{\alpha} \partial_y h(t)\|_{L^2_{l+k}(\Omega)} + \delta_1 \mu \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 \\ & \quad + \delta_1 \nu \|\partial_y h(t)\|_{\mathcal{H}_0^m}^2 + \delta_1^{-1} (\|(u, \theta, h)(t)\|_{\mathcal{H}_l^m}^8 + 1) + \sum_{|\beta| \leq m+2} \|\partial_{\tau}^{\beta}(U, \Theta, H)(t)\|_{L^2(\mathbb{T}_x)}^8. \end{aligned} \quad (3.82)$$

Similarly(or see [42]), it is easy to deduce that

$$\begin{aligned} & - \int_{\Omega} D^{\alpha} I_1 \cdot \langle y \rangle^{2k+2l} D^{\alpha} u dx dy - \int_{\Omega} D^{\alpha} I_3 \cdot \langle y \rangle^{2k+2l} D^{\alpha} h dx dy \\ & \leq \frac{\kappa}{4} \|D^{\alpha} \partial_y \theta(t)\|_{L^2_{k+l}(\Omega)}^2 + C(\|(u, \theta, h)(t)\|_{\mathcal{H}_l^m}^4 + 1) + \sum_{|\beta| \leq m+2} \|\partial_{\tau}^{\beta}(U, \Theta, H)(t)\|_{L^2(\mathbb{T}_x)}^4. \end{aligned} \quad (3.83)$$

Therefore, the combination of (3.82) and (3.83) completes the proof of (3.15). \square

3.3. Weighted H_l^m -estimates only in tangential variable.

Similar to the classical Prandtl equations, an essential difficulty for solving the problem (2.6) arises from the loss of one derivative in the tangential variable x in the terms $v \partial_y u - g \partial_y h, v \partial_y h - g \partial_y u$ and $v \partial_y \theta$. More precisely, we recall the following nonlinear MHD boundary layer equations

$$\begin{cases} \partial_t u + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h \\ \quad - \mu \partial_y[(\theta + \Theta\phi'(y) + 1)\partial_y u] + U\phi''v - H\phi''g + II_1 = r_1, \\ c_v\{\partial_t \theta + (u + U\phi')\partial_x \theta + (v - U_x\phi)\partial_y \theta\} - \kappa \partial_y^2 \theta + c_v \Theta \phi''v + II_2 = r_2, \\ \partial_t h + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u - \nu \partial_y^2 h \\ \quad + H\phi''v - U\phi''g + II_3 = r_3, \\ \partial_x u + \partial_y v = 0, \quad \partial_x h + \partial_y g = 0, \end{cases} \quad (3.84)$$

where

$$\begin{aligned} II_1 & \triangleq U_x \phi' u - H_x \phi' h - U \phi^{(3)} \theta - U \phi'' \theta_y, \quad II_3 \triangleq H_x \phi' u - U_x \phi' h, \\ II_2 & \triangleq c_v \Theta_x \phi' u - \mu \theta (u_y)^2 - \mu (U \phi'')^2 \theta - 2\mu U \phi'' \theta u_y - \mu \Theta \phi' (u_y)^2 \\ & \quad - 2\mu \Theta U \phi' \phi'' u_y - 2\mu U \phi'' u_y - \mu (u_y)^2 - \nu (h_y)^2 - 2\nu H \phi' h_y. \end{aligned}$$

Then, applying β -th ($|\beta| = m$) order tangential derivatives on the equations (3.84), we find

$$\begin{cases} \partial_t \partial_{\tau}^{\beta} u + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y] \partial_{\tau}^{\beta} u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y] \partial_{\tau}^{\beta} h \\ \quad + (\partial_y u + U\phi'') \partial_{\tau}^{\beta} v - (\partial_y h + H\phi'') \partial_{\tau}^{\beta} g - \mu \partial_y[(\theta + \Theta\phi' + 1)\partial_y \partial_{\tau}^{\beta} u] = \partial_{\tau}^{\beta}(r_1 - II_1) + R_u^{\beta}, \\ c_v[\partial_t + (u + U\phi')\partial_x + (v - U_x\phi)\partial_y] \partial_{\tau}^{\beta} \theta - \kappa \partial_y^2 \partial_{\tau}^{\beta} \theta + c_v(\partial_y \theta + \Theta\phi'') \partial_{\tau}^{\beta} v = \partial_{\tau}^{\beta}(r_2 - II_2) + R_{\theta}^{\beta}, \\ \partial_t \partial_{\tau}^{\beta} h + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y] \partial_{\tau}^{\beta} h - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y] \partial_{\tau}^{\beta} u \\ \quad + (\partial_y h + H\phi'') \partial_{\tau}^{\beta} v - (\partial_y u + U\phi'') \partial_{\tau}^{\beta} g - \nu \partial_y^2 \partial_{\tau}^{\beta} h = \partial_{\tau}^{\beta}(r_3 - II_3) + R_h^{\beta}, \end{cases} \quad (3.85)$$

where

$$\begin{aligned}
 R_u^\beta &\triangleq -[\partial_\tau^\beta, (u + U\phi')\partial_x - U_x\phi\partial_y]u + [\partial_\tau^\beta, (h + H\phi')\partial_x - H_x\phi\partial_y]h \\
 &\quad - [\partial_\tau^\beta, U\phi'']v + [\partial_\tau^\beta, H\phi'']g + \mu\partial_y([\partial_\tau^\beta, (\theta + \Theta\phi' + 1)\partial_y]u) \\
 &\quad - \sum_{0 < \tilde{\beta} < \beta} C_{\tilde{\beta}}^\beta \partial_\tau^\beta v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y u + \sum_{0 < \tilde{\beta} < \beta} C_{\tilde{\beta}}^\beta \partial_\tau^\beta g \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y h, \\
 R_\theta^\beta &\triangleq -c_v[\partial_\tau^\beta, (u + U\phi')\partial_x - U_x\phi\partial_y]\theta - c_v[\partial_\tau^\beta, \Theta\phi'']v - \sum_{0 < \tilde{\beta} < \beta} c_v C_{\tilde{\beta}}^\beta \partial_\tau^\beta v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y \theta, \\
 R_h^\beta &\triangleq -[\partial_\tau^\beta, (u + U\phi')\partial_x - U_x\phi\partial_y]h + [\partial_\tau^\beta, (h + H\phi')\partial_x - H_x\phi\partial_y]u - [\partial_\tau^\beta, H\phi'']v \\
 &\quad + [\partial_\tau^\beta, U\phi'']g - \sum_{0 < \tilde{\beta} < \beta} C_{\tilde{\beta}}^\beta \partial_\tau^\beta v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y h + \sum_{0 < \tilde{\beta} < \beta} C_{\tilde{\beta}}^\beta \partial_\tau^\beta g \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y u.
 \end{aligned} \tag{3.86}$$

On the other hand, similar to [42], it is easy to verify that the function $\partial_y^{-1}h$ satisfies

$$\partial_t(\partial_y^{-1}h) + (v - U_x\phi)(h + H\phi') - (g - H_x\phi)(u + U\phi') - \nu\partial_y h = -H_t\phi + \nu H\phi'', \tag{3.87}$$

or equivalently

$$\partial_t(\partial_y^{-1}h) + (h + H\phi')v + (u + U\phi')\partial_x(\partial_y^{-1}h) - U_x\phi h + H_x\phi u - \nu\partial_y h = H_t\phi(\phi' - 1) + \nu H\phi''. \tag{3.88}$$

In view of the divergence free condition $\partial_x h + \partial_y g = 0$, then there exists a stream function ψ such that

$$h = \partial_y \psi, \quad g = -\partial_x \psi, \quad \psi|_{y=0} = 0. \tag{3.89}$$

Then, the combination of (3.88) and (3.89) implies that the function ψ satisfies the following equation

$$\partial_t \psi + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\psi - \nu\partial_y^2 \psi + H\phi'v + H_x\phi u = r_4, \tag{3.90}$$

where

$$r_4 \triangleq H_t\phi(1 - \phi') + \nu H\phi''. \tag{3.91}$$

Then applying m -th order tangential spatial derivative to the equation (3.90), we find

$$\partial_t \partial_\tau^\beta \psi + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\partial_\tau^\beta \psi + (h + H\phi')\partial_\tau^\beta v - \nu\partial_y^2 \partial_\tau^\beta \psi = \partial_\tau^\beta r_4 + R_4^\beta, \tag{3.92}$$

where

$$R_4^\beta \triangleq -\partial_\tau^\beta(H_x\phi u) - [\partial_\tau^\beta, H\phi']v - [\partial_\tau^\beta, (u + U\phi')\partial_x - U_x\phi\partial_y]\psi - \sum_{0 < \tilde{\beta} < \beta} C_{\tilde{\beta}}^\beta \partial_\tau^\beta v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y \psi. \tag{3.93}$$

Let us define the functions

$$u_\beta \triangleq \partial_\tau^\beta u - \eta_1 \partial_\tau^\beta \psi, \quad \theta_\beta \triangleq \partial_\tau^\beta \theta - \eta_2 \partial_\tau^\beta \psi, \quad h_\beta \triangleq \partial_\tau^\beta h - \eta_3 \partial_\tau^\beta \psi, \tag{3.94}$$

where

$$\eta_1 \triangleq \frac{\partial_y u + U\phi''}{h + H\phi'}, \quad \eta_2 \triangleq \frac{\partial_y \theta + \Theta\phi''}{h + H\phi'}, \quad \eta_3 \triangleq \frac{\partial_y h + H\phi''}{h + H\phi'}.$$

Then, we can obtain the following estimates:

$$M(t)^{-1} \|\partial_\tau^\beta(u, \theta, h)(t)\|_{L_t^2(\Omega)} \leq \|(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_t^2(\Omega)} \leq M(t) \|\partial_\tau^\beta(u, \theta, h)(t)\|_{L_t^2(\Omega)} \tag{3.95}$$

and

$$\|\partial_y \partial_\tau^\beta(u, \theta, h)(t)\|_{L_t^2(\Omega)} \leq \|\partial_y(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_t^2(\Omega)} + M(t) \|h_\beta(t)\|_{L_t^2(\Omega)}, \tag{3.96}$$

where

$$\begin{aligned} M(t) \triangleq & 2\delta_0^{-1}(C\|(U, \Theta, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y \rangle^{l+1} \partial_y(u, \theta, h)(t)\|_{L^\infty(\Omega)}) \\ & + 2\delta_0^{-1}(\|\langle y \rangle^{l+1} \partial_y^2(u, \theta, h)(t)\|_{L^\infty(\Omega)} + 1). \end{aligned} \quad (3.97)$$

The detail of proof for the estimates (3.95) and (3.96) can be found in Appendix B. On the other hand, we know from the assumption (3.1) that

$$\|\langle y \rangle^{l+1} \partial_y^i(u, \theta, h)(t)\|_{L^\infty(\Omega)} \leq C\delta_0^{-1}, \quad \text{for } i = 1, 2, \quad t \in [0, T]. \quad (3.98)$$

Then, one can get, for δ_0 sufficiently small, that

$$M(t) \leq 2\delta_0^{-1}(C\|(U, \Theta, H)(t)\|_{L^\infty(\mathbb{T}_x)} + 2\delta_0^{-1} + 1) \leq 6\delta_0^{-2}. \quad (3.99)$$

Therefore, we deduce from the definition of functions (3.94), and equations (3.85) and (3.92) that

$$\begin{cases} \partial_t u_\beta + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u_\beta - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h_\beta + \nu\eta_1\partial_y h_\beta \\ \quad - \mu\partial_y[(\theta + \Theta\phi' + 1)\partial_y u_\beta] - \mu\partial_y[(\theta + \Theta\phi' + 1)(\partial_y\eta_1\partial_\tau^\beta\psi + \eta_1\partial_\tau^\beta h)] = R_1^\beta, \\ c_v[\partial_t + (u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\theta_\beta - \kappa\partial_y^2\theta_\beta - \kappa\partial_y(\partial_y\eta_2\partial_\tau^\beta\psi + \eta_2\partial_\tau^\beta h) + c_v\nu\eta_2\partial_y h_\beta = R_2^\beta, \\ \partial_t h_\beta + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h_\beta - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u_\beta - \nu\partial_y^2 h_\beta = R_3^\beta, \end{cases} \quad (3.100)$$

where

$$\begin{cases} R_1^\beta \triangleq \partial_\tau^\beta(r_1 - II_1) + R_u^\beta - \eta_1\partial_\tau^\beta r_4 - \eta_1 R_4^\beta - \nu\eta_1\eta_3\partial_\tau^\beta h + \eta_3(g - H_x\phi)\partial_\tau^\beta h - \zeta_1(\partial_\tau^\beta\psi), \\ R_2^\beta \triangleq \partial_\tau^\beta(r_2 - II_2) + R_\theta^\beta - c_v\eta_2\partial_\tau^\beta r_4 - c_v\eta_2 R_4^\beta - c_v\nu\eta_2\eta_3\partial_\tau^\beta h - c_v\zeta_2(\partial_\tau^\beta\psi), \\ R_3^\beta \triangleq \partial_\tau^\beta(r_3 - II_3) + R_h^\beta - \eta_3\partial_\tau^\beta r_4 - \eta_3 R_4^\beta + [2\nu\partial_y\eta_3 + (g - H_x\phi)\eta_1]\partial_\tau^\beta h - \zeta_3(\partial_\tau^\beta\psi), \end{cases} \quad (3.101)$$

with

$$\begin{cases} \zeta_1 \triangleq \partial_t\eta_1 + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\eta_1 - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]\eta_3 + \nu\eta_1\partial_y\eta_3, \\ \zeta_2 \triangleq \partial_t\eta_2 + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\eta_2 + \nu\eta_2\partial_y\eta_3, \\ \zeta_3 \triangleq \partial_t\eta_3 + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\eta_3 - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]\eta_1 - \nu\partial_y^2\eta_3. \end{cases} \quad (3.102)$$

Also, we have the corresponding initial and boundary conditions as follows:

$$\begin{cases} u_\beta|_{t=0} = \partial_\tau^\beta u(0, x, y) - \frac{\partial_y u_0(x, y) + U(0, x)\phi''(y)}{h_0(x, y) + H(0, x)\phi'(y)} \int_0^y \partial_\tau^\beta h(0, x, z) dz \triangleq u_{\beta 0}(x, y), \\ \theta_\beta|_{t=0} = \partial_\tau^\beta \theta(0, x, y) - \frac{\partial_y \theta_0(x, y) + \Theta(0, x)\phi''(y)}{h_0(x, y) + H(0, x)\phi'(y)} \int_0^y \partial_\tau^\beta h(0, x, z) dz \triangleq \theta_{\beta 0}(x, y), \\ h_\beta|_{t=0} = \partial_\tau^\beta h(0, x, y) - \frac{\partial_y h_0(x, y) + H(0, x)\phi''(y)}{h_0(x, y) + H(0, x)\phi'(y)} \int_0^y \partial_\tau^\beta h(0, x, z) dz \triangleq h_{\beta 0}(x, y), \\ u_\beta|_{y=0} = \theta_\beta|_{y=0} = \partial_y h_\beta|_{y=0} = 0. \end{cases} \quad (3.103)$$

Moreover, by combining $\psi = \partial_y^{-1}h$ with the inequality (A.3), it is easy to check that

$$\|\langle y \rangle^{-1} \partial_\tau^\beta \psi(t)\|_{L^2(\Omega)} \leq 2\|\partial_\tau^\beta h(t)\|_{L^2(\Omega)}. \quad (3.104)$$

By virtue of Sobolev embedding inequality and direct computation, we have for any $\lambda \in \mathbb{R}$ and $i = 1, 2, 3$,

$$\begin{aligned} \|\langle y \rangle^\lambda \eta_i\|_{L^\infty(\Omega)} & \leq C\delta_0^{-1}(\|(U, \Theta, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u, \theta, h)(t)\|_{\mathcal{H}_{\lambda-1}^3}), \\ \|\langle y \rangle^\lambda \partial_y \eta_i\|_{L^\infty(\Omega)} & \leq C\delta_0^{-2}(\|(U, \Theta, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u, \theta, h)(t)\|_{\mathcal{H}_{\lambda-1}^4})^2, \\ \|\langle y \rangle^\lambda \zeta_i\|_{L^\infty(\Omega)} & \leq C\delta_0^{-3} \left(\sum_{|\beta| \leq 1} \|\partial_\tau^\beta (U, \Theta, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u, \theta, h)(t)\|_{\mathcal{H}_{\lambda-1}^5} \right)^3. \end{aligned} \quad (3.105)$$

Now, we are going to establish the L_t^2 -norms for the quantity $(u_\beta, \theta_\beta, h_\beta)$.

Lemma 3.4. *[L_t^2 -estimate on $(u_\beta, \theta_\beta, h_\beta)$] Under the hypotheses of Proposition 3.1, we have for any $t \in [0, T]$ and the quantity $(u_\beta, \theta_\beta, h_\beta)$ given in (3.94) that*

$$\begin{aligned} & \sum_{|\beta|=m} \left\{ \frac{d}{dt} \left(\| (u_\beta, h_\beta)(t) \|_{L_t^2(\Omega)}^2 + c_v \| \theta_\beta(t) \|_{L_t^2(\Omega)}^2 \right) + c_0 \| \partial_y (u_\beta, \theta_\beta, h_\beta)(t) \|_{L_t^2(\Omega)}^2 \right\} \\ & \leq C \delta_0^{-4} \left(1 + \sum_{|\beta| \leq m+2} \| \partial_\tau^\beta (U, \Theta, H)(t) \|_{L^2(\mathbb{T}_x)}^{10} \right) + C \delta_0^{-4} \| (u_\beta, \theta_\beta, h_\beta)(t) \|_{L_t^2(\Omega)}^4 \\ & \quad + C \delta_0^{-4} \| (u, \theta, h)(t) \|_{\mathcal{H}_t^m}^{12} + \sum_{|\beta|=m} \left\{ \sum_{i=1}^3 \| \partial_\tau^\beta r_i \|_{L_t^2(\Omega)}^2 + \| \eta_i \partial_\tau^\beta r_4 \|_{L_t^2(\Omega)}^2 \right\}, \end{aligned} \quad (3.106)$$

where the quantity c_0 is defined in Lemma 3.3.

Proof. Multiplying the equation (3.100)₁ by $\langle y \rangle^{2l} u_\beta$, integrating over Ω and integrating by part, we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega \langle y \rangle^{2l} |u_\beta|^2 dx dy + \mu \int_\Omega (\theta + \Theta \phi' + 1) \langle y \rangle^{2l} |\partial_y u_\beta|^2 dx dy \\ & = - \int_\Omega [(h + H \phi') \partial_x + (g - H_x \phi) \partial_y] u_\beta \cdot \langle y \rangle^{2l} h_\beta dx dy + l \int_\Omega (v - U_x \phi) \cdot \langle y \rangle^{2l-1} |u_\beta|^2 dx dy \\ & \quad - \nu \int_\Omega \eta_1 \partial_y h_\beta \cdot \langle y \rangle^{2l} u_\beta dx dy - 2l \int_\Omega (g - H_x \phi) \cdot \langle y \rangle^{2l-1} u_\beta h_\beta dx dy \\ & \quad - \mu \int_\Omega (\theta + \Theta \phi' + 1) (\partial_y \eta_1 \partial_\tau^\beta \psi + \eta_1 \partial_\tau^\beta h) \cdot \langle y \rangle^{2l} \partial_y u_\beta dx dy \\ & \quad - 2l \mu \int_\Omega (\theta + \Theta \phi' + 1) (\partial_y \eta_1 \partial_\tau^\beta \psi + \eta_1 \partial_\tau^\beta h) \cdot \langle y \rangle^{2l-1} u_\beta dx dy \\ & \quad - 2l \mu \int_\Omega (\theta + \Theta \phi' + 1) \partial_y u_\beta \cdot \langle y \rangle^{2l-1} u_\beta dx dy + \int_\Omega R_1^\beta \cdot \langle y \rangle^{2l} u_\beta dx dy. \end{aligned} \quad (3.107)$$

In view of the inequality (A.5), Sobolev and Cauchy inequalities, it is easy to deduce that

$$\begin{aligned} & \left| l \int_\Omega (v - U_x \phi) \langle y \rangle^{2l-1} |u_\beta|^2 dx dy \right| \\ & \leq C (\| \langle y \rangle^{-1} \partial_y^{-1} u_x \|_{L^\infty(\Omega)} + \| \langle y \rangle^{-1} (U_x \phi) \|_{L^\infty(\Omega)}) \| u_\beta \|_{L_t^2(\Omega)}^2 \\ & \leq C (\| u_x \|_{L^\infty(\Omega)} + \| U_x \|_{L^\infty(\mathbb{T}_x)}) \| u_\beta \|_{L_t^2(\Omega)}^2 \\ & \leq C (1 + \| U_x \|_{H^1(\mathbb{T}_x)}^2) + C (\| u \|_{\mathcal{H}_t^m}^2 + \| u_\beta \|_{L_t^2(\Omega)}^4). \end{aligned} \quad (3.108)$$

Similarly, we also find that

$$\left| -2l \int_\Omega (g - H_x \phi) \langle y \rangle^{2l-1} u_\beta h_\beta dx dy \right| \leq C (1 + \| H_x \|_{H^1(\mathbb{T}_x)}^2) + C (\| h \|_{\mathcal{H}_t^m}^2 + \| (u_\beta, h_\beta) \|_{L_t^2(\Omega)}^4). \quad (3.109)$$

By virtue of the Cauchy inequality and the estimate (3.105), one arrives at directly

$$\begin{aligned} & \left| -\nu \int_\Omega \eta_1 \partial_y h_\beta \langle y \rangle^{2l} u_\beta dx dy \right| \\ & \leq \frac{\nu}{12} \| \partial_y h_\beta \|_{L_t^2(\Omega)}^2 + C \| \eta_1 \|_{L^\infty(\Omega)}^2 \| u_\beta \|_{L_t^2(\Omega)}^2 \\ & \leq \frac{\nu}{12} \| \partial_y h_\beta \|_{L_t^2(\Omega)}^2 + C \delta_0^{-2} (\| (U, H) \|_{H^1(\mathbb{T}_x)}^2 + \| (u, h) \|_{\mathcal{H}_0^3}^2) \| u_\beta \|_{L_t^2(\Omega)}^2 \\ & \leq \frac{\nu}{12} \| \partial_y h_\beta \|_{L_t^2(\Omega)}^2 + C \delta_0^{-2} \| (U, H) \|_{H^1(\mathbb{T}_x)}^4 + C \delta_0^{-2} (\| (u, h) \|_{\mathcal{H}_t^m}^4 + \| u_\beta \|_{L_t^2(\Omega)}^4). \end{aligned} \quad (3.110)$$

With the help of estimate (3.105), Hölder and Cauchy inequalities, it is easy to check that

$$\begin{aligned}
& \left| -\mu \int_{\Omega} (\theta + \Theta\phi' + 1)(\partial_y \eta_1 \partial_{\tau}^{\beta} \psi + \eta_1 \partial_{\tau}^{\beta} h) \langle y \rangle^{2l} \partial_y u_{\beta} dx dy \right| \\
& \leq \mu \|\partial_y u_{\beta}\|_{L_t^2(\Omega)} \|\theta + \Theta\phi' + 1\|_{L^{\infty}(\Omega)} \|\langle y \rangle^{l+1} \partial_y \eta_1\|_{L^{\infty}(\Omega)} \|\langle y \rangle^{-1} \partial_{\tau}^{\beta} \psi\|_{L^2(\Omega)} \\
& \quad + \mu \|\partial_y u_{\beta}\|_{L_t^2(\Omega)} \|\theta + \Theta\phi' + 1\|_{L^{\infty}(\Omega)} \|\eta_1\|_{L^{\infty}(\Omega)} \|\partial_{\tau}^{\beta} h\|_{L_t^2(\Omega)} \\
& \leq C\delta_0^{-4} (\|\theta\|_{H^2(\Omega)} + \|\Theta\|_{H^1(\mathbb{T}_x)} + 1)^2 (\|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_t^4})^4 \|\partial_{\tau}^{\beta} h\|_{L^2(\Omega)}^2 \\
& \quad + C\delta_0^{-2} (\|\theta\|_{H^2(\Omega)} + \|\Theta\|_{H^1(\mathbb{T}_x)} + 1)^2 (\|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_t^3})^2 \|\partial_{\tau}^{\beta} h\|_{L^2(\Omega)}^2 \\
& \quad + \frac{\mu}{12} \|\partial_y u_{\beta}\|_{L_t^2(\Omega)}^2 \\
& \leq \frac{\mu}{12} \|\partial_y u_{\beta}\|_{L_t^2(\Omega)}^2 + C\delta_0^{-4} (1 + \|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)}^8) + C\delta_0^{-4} \|(u, \theta, h)\|_{\mathcal{H}_t^m}^2.
\end{aligned} \tag{3.111}$$

Similarly, we can obtain the following estimates

$$\begin{aligned}
& \left| -2l\mu \int_{\Omega} (\theta + \Theta\phi' + 1)(\partial_y \eta_1 \partial_{\tau}^{\beta} \psi + \eta_1 \partial_{\tau}^{\beta} h) \langle y \rangle^{2l-1} u_{\beta} dx dy \right| \\
& \leq C(\|\theta\|_{L^{\infty}(\Omega)} + \|\Theta\|_{L^{\infty}(\mathbb{T}_x)} + 1) \|\langle y \rangle^l \partial_y \eta_1\|_{L^{\infty}(\Omega)} \|\langle y \rangle^{-1} \partial_{\tau}^{\beta} \psi\|_{L^2(\Omega)} \|u_{\beta}\|_{L_t^2(\Omega)} \\
& \quad + C(\|\theta\|_{L^{\infty}(\Omega)} + \|\Theta\|_{L^{\infty}(\mathbb{T}_x)} + 1) \|\eta_1\|_{L^{\infty}(\Omega)} \|\partial_{\tau}^{\beta} h\|_{L_t^2(\Omega)} \|u_{\beta}\|_{L_t^2(\Omega)} \\
& \leq C\delta_0^{-2} (\|\theta\|_{H^2(\Omega)} + \|\Theta\|_{H^1(\mathbb{T}_x)} + 1) (\|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_t^4})^2 \|\partial_{\tau}^{\beta} h\|_{L^2(\Omega)} \|u_{\beta}\|_{L_t^2(\Omega)} \\
& \quad + C\delta_0^{-1} (\|\theta\|_{H^2(\Omega)} + \|\Theta\|_{H^1(\mathbb{T}_x)} + 1) (\|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_t^3}) \|\partial_{\tau}^{\beta} h\|_{L^2(\Omega)} \|u_{\beta}\|_{L_t^2(\Omega)} \\
& \leq C\delta_0^{-2} (1 + \|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)}^{10}) + C\delta_0^{-2} (\|(u, \theta, h)\|_{\mathcal{H}_t^m}^8 + \|u_{\beta}\|_{L_t^2(\Omega)}^4),
\end{aligned} \tag{3.112}$$

and

$$\begin{aligned}
& \left| -2l\mu \int_{\Omega} (\theta + \Theta\phi' + 1) \partial_y u_{\beta} \cdot \langle y \rangle^{2l-1} u_{\beta} dx dy \right| \\
& \leq C\mu \|\partial_y u_{\beta}\|_{L_t^2(\Omega)} \|\theta + \Theta\phi' + 1\|_{L^{\infty}(\Omega)} \|u_{\beta}\|_{L_t^2(\Omega)} \\
& \leq \frac{\mu}{12} \|\partial_y u_{\beta}\|_{L_t^2(\Omega)}^2 + C(1 + \|\Theta\|_{H^1(\mathbb{T}_x)}^2)^2 + C(\|\theta\|_{\mathcal{H}_t^m}^4 + \|u_{\beta}\|_{L_t^2(\Omega)}^4).
\end{aligned} \tag{3.113}$$

By virtue of the lower bound estimate for temperature (3.5), we get

$$\mu \int_{\Omega} (\theta + \Theta\phi' + 1) \langle y \rangle^{2l} |\partial_y u_{\beta}|^2 dx dy \geq \mu \int_{\Omega} \langle y \rangle^{2l} |\partial_y u_{\beta}|^2 dx dy. \tag{3.114}$$

Plugging the estimates (3.108)-(3.114) into (3.107), it is easy to deduce that

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \langle y \rangle^{2l} |u_{\beta}|^2 dx dy + \mu \int_{\Omega} \langle y \rangle^{2l} |\partial_y u_{\beta}|^2 dx dy \\
& \leq - \int_{\Omega} [(h + H\phi') \partial_x + (g - H_x \phi) \partial_y] u_{\beta} \cdot \langle y \rangle^{2l} h_{\beta} dx dy \\
& \quad + \frac{\nu}{12} \|\partial_y h_{\beta}\|_{L_t^2(\Omega)}^2 + C\delta_0^{-4} (1 + \|(U, \Theta, H)\|_{H^2(\mathbb{T}_x)}^{10}) \\
& \quad + C\delta_0^{-4} (\|(u, \theta, h)\|_{\mathcal{H}_t^m}^2 + \|(u_{\beta}, h_{\beta})\|_{L_t^2(\Omega)}^4) + \int_{\Omega} R_1^{\beta} \cdot \langle y \rangle^{2l} u_{\beta} dx dy.
\end{aligned} \tag{3.115}$$

Multiplying equation (3.100)₂ by $\langle y \rangle^{2l} h_\beta$, integrating over Ω and integrating by part, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \langle y \rangle^{2l} |h_\beta|^2 dx dy + \nu \int_{\Omega} \langle y \rangle^{2l} |\partial_y h_\beta|^2 dx dy \\ &= l \int_{\Omega} (v - U_x \phi) \langle y \rangle^{2l-1} |h_\beta|^2 dx dy - 2l\nu \int_{\Omega} \langle y \rangle^{2l-1} \partial_y h_\beta \cdot h_\beta dx dy \\ & \quad + \int_{\Omega} [(h + H\phi') \partial_x + (g - H_x \phi) \partial_y] u_\beta \cdot \langle y \rangle^{2l} h_\beta dx dy + \int_{\Omega} R_3^\beta \cdot \langle y \rangle^{2l} h_\beta dx dy. \end{aligned} \quad (3.116)$$

Similar to the estimate (3.108), we can obtain directly

$$\left| c_v l \int_{\Omega} (v - U_x \phi) \langle y \rangle^{2l-1} |h_\beta|^2 dx dy \right| \leq C \|U_x\|_{H^1(\mathbb{T}_x)}^2 + C(\|u\|_{\mathcal{H}_l^m}^2 + \|h_\beta\|_{L_l^2(\Omega)}^4). \quad (3.117)$$

In view of the Cauchy inequality, it is easy to deduce that

$$\left| 2l\nu \int_{\Omega} \langle y \rangle^{2l-1} \partial_y h_\beta \cdot h_\beta dx dy \right| \leq \frac{\nu}{2} \|\partial_y h_\beta\|_{L_l^2(\Omega)}^2 + C \|h_\beta\|_{L_l^2(\Omega)}^2. \quad (3.118)$$

Substituting (3.117) and (3.118) into (3.116), we find

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \langle y \rangle^{2l} |h_\beta|^2 dx dy + \nu \int_{\Omega} \langle y \rangle^{2l} |\partial_y h_\beta|^2 dx dy \\ & \leq \int_{\Omega} [(h + H\phi') \partial_x + (g - H_x \phi) \partial_y] u_\beta \cdot \langle y \rangle^{2l} h_\beta dx dy + \int_{\Omega} R_3^\beta \cdot \langle y \rangle^{2l} h_\beta dx dy \\ & \quad + C(1 + \|U_x\|_{H^1(\mathbb{T}_x)}^2) + C(\|u\|_{\mathcal{H}_l^m}^2 + \|h_\beta\|_{L_l^2(\Omega)}^4), \end{aligned}$$

which, together with the inequality (3.115), implies that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \langle y \rangle^{2l} (|u_\beta|^2 + |h_\beta|^2) dx dy + \mu \int_{\Omega} \langle y \rangle^{2l} |\partial_y u_\beta|^2 dx dy + \nu \int_{\Omega} \langle y \rangle^{2l} |\partial_y h_\beta|^2 dx dy \\ & \leq C\delta_0^{-4} (1 + \|(U, \Theta, H)(t)\|_{H^2(\mathbb{T}_x)}^{10}) + C\delta_0^{-4} (\|(u, \theta, h)(t)\|_{\mathcal{H}_l^m}^{12} + \|(u_\beta, h_\beta)(t)\|_{L_l^2(\Omega)}^4) \\ & \quad + \int_{\Omega} R_1^\beta \cdot \langle y \rangle^{2l} u_\beta dx dy + \int_{\Omega} R_3^\beta \cdot \langle y \rangle^{2l} h_\beta dx dy. \end{aligned} \quad (3.119)$$

Multiplying equation (3.100)₃ by $\langle y \rangle^{2l} \theta_\beta$, integrating over Ω and integrating by part, we find

$$\begin{aligned} & \frac{c_v}{2} \frac{d}{dt} \int_{\Omega} \langle y \rangle^{2l} |\theta_\beta|^2 dx dy + \frac{\kappa}{2} \int_{\Omega} \langle y \rangle^{2l} |\partial_y \theta_\beta|^2 dx dy \\ &= c_v l \int_{\Omega} (v - U_x \phi) \langle y \rangle^{2l-1} |\theta_\beta|^2 dx dy - 2\kappa l \int_{\Omega} \langle y \rangle^{2l-1} \partial_y \theta_\beta \cdot \theta_\beta dx dy \\ & \quad - 2\kappa l \int_{\Omega} (\partial_y \eta_2 \partial_\tau^\beta \psi + \eta_2 \partial_\tau^\beta h) \cdot \langle y \rangle^{2l-1} \theta_\beta dx dy - c_v \nu \int_{\Omega} \eta_2 \partial_y h_\beta \cdot \langle y \rangle^{2l} \theta_\beta dx dy \\ & \quad - \kappa \int_{\Omega} (\partial_y \eta_2 \partial_\tau^\beta \psi + \eta_2 \partial_\tau^\beta h) \cdot \langle y \rangle^{2l} \theta_\beta dx dy + \int_{\Omega} R_2^\beta \cdot \langle y \rangle^{2l} \theta_\beta dx dy. \end{aligned} \quad (3.120)$$

Similar to the estimates (3.108) and (3.110), we can obtain directly

$$\begin{aligned} & \left| c_v l \int_{\Omega} (v - U_x \phi) \langle y \rangle^{2l-1} |\theta_\beta|^2 dx dy \right| \leq C \|U_x\|_{H^1(\mathbb{T}_x)}^2 + C(\|u_x\|_{H^2(\Omega)}^2 + \|\theta_\beta\|_{L_l^2(\Omega)}^4), \\ & \left| c_v \nu \int_{\Omega} \eta_2 \partial_y h_\beta \cdot \langle y \rangle^{2l} \theta_\beta dx dy \right| \leq \frac{\nu}{2} \|\partial_y h_\beta\|_{L_l^2(\Omega)}^2 + C\delta_0^{-4} (\|(U, H)\|_{H^1(\mathbb{T}_x)}^4 + \|(u, h)\|_{\mathcal{H}_l^m}^4 + \|\theta_\beta\|_{L_l^2(\Omega)}^4). \end{aligned} \quad (3.121)$$

With the help of Cauchy inequality, one arrives at immediately

$$\left| 2\kappa l \int \langle y \rangle^{2l-1} \partial_y \theta_\beta \cdot \theta_\beta dx dy \right| \leq \frac{\kappa}{2} \|\partial_y \theta_\beta\|_{L_t^2(\Omega)}^2 + C \|\theta_\beta\|_{L_t^2(\Omega)}^2. \quad (3.122)$$

On the other hand, we apply the Hölder inequality and estimate (3.105) to get that

$$\begin{aligned} & \left| -2\kappa l \int (\partial_y \eta_2 \partial_\tau^\beta \psi + \eta_2 \partial_\tau^\beta h) \cdot \langle y \rangle^{2l-1} \theta_\beta dx dy \right| \\ & \leq C \|\langle y \rangle^l \partial_y \eta_2\|_{L^\infty(\Omega)} \|\langle y \rangle^{-1} \partial_\tau^\beta \psi\|_{L^2(\Omega)} \|\langle y \rangle^l \theta_\beta\|_{L^2(\Omega)} \\ & \quad + C \|\eta_2\|_{L^\infty(\Omega)} \|\partial_\tau^\beta h\|_{L_t^2(\Omega)} \|\theta_\beta\|_{L_t^2(\Omega)} \\ & \leq C \delta_0^{-2} (\|(U, \Theta, h)\|_{L^\infty(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_l^4})^2 \|\partial_\tau^\beta h\|_{L^2(\Omega)} \|\theta_\beta\|_{L_t^2(\Omega)} \\ & \quad + C \delta_0^{-1} (\|(U, \Theta, h)\|_{L^\infty(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_l^3}) \|\partial_\tau^\beta h\|_{L^2(\Omega)} \|\theta_\beta\|_{L_t^2(\Omega)} \\ & \leq C \delta_0^{-2} (1 + \|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)}^8) + C \delta_0^{-2} (\|(u, \theta, h)\|_{\mathcal{H}_l^m}^8 + \|\theta_\beta\|_{L_t^2(\Omega)}^2), \end{aligned} \quad (3.123)$$

and

$$\begin{aligned} & \left| -\kappa \int (\partial_y \eta_2 \partial_\tau^\beta \psi + \eta_2 \partial_\tau^\beta h) \cdot \langle y \rangle^{2l} \theta_\beta dx dy \right| \\ & \leq \|\langle y \rangle^{l+1} \partial_y \eta_2\|_{L^\infty(\Omega)} \|\langle y \rangle^{-1} \partial_\tau^\beta \psi\|_{L^2(\Omega)} \|\theta_\beta\|_{L_t^2(\Omega)} \\ & \quad + \|\eta_2\|_{L^\infty(\Omega)} \|\partial_\tau^\beta h\|_{L_t^2(\Omega)} \|\theta_\beta\|_{L_t^2(\Omega)} \\ & \leq C \delta_0^{-2} (\|(U, \Theta, H)\|_{L^\infty(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_l^4})^2 \|\partial_\tau^\beta h\|_{L^2(\Omega)} \|\theta_\beta\|_{L_t^2(\Omega)} \\ & \quad + C \delta_0^{-1} (\|(U, \Theta, H)\|_{L^\infty(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_l^3}) \|\partial_\tau^\beta h\|_{L^2(\Omega)} \|\theta_\beta\|_{L_t^2(\Omega)} \\ & \leq C \delta_0^{-2} (1 + \|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)}^8) + C \delta_0^{-2} (\|(u, \theta, h)\|_{\mathcal{H}_l^m}^8 + \|\theta_\beta\|_{L_t^2(\Omega)}^2). \end{aligned} \quad (3.124)$$

Substituting the estimates (3.121)-(3.124) into (3.120), we find immediately

$$\begin{aligned} & \frac{d}{dt} \int_\Omega \langle y \rangle^{2l} |\theta_\beta|^2 dx dy + \kappa \int_\Omega \langle y \rangle^{2l} |\partial_y \theta_\beta|^2 dx dy \\ & \leq C \delta_0^{-4} (1 + \|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)}^8) + C \delta_0^{-4} (\|(u, \theta, h)\|_{\mathcal{H}_l^m}^8 + \|\theta_\beta\|_{L_t^2(\Omega)}^4) \\ & \quad + \frac{\nu}{2} \|\partial_y h_\beta\|_{L_t^2(\Omega)}^2 + \int_\Omega R_2^\beta \cdot \langle y \rangle^{2l} \theta^\beta dx dy. \end{aligned} \quad (3.125)$$

The combination of (3.119) and (3.125) yields directly

$$\begin{aligned} & \frac{d}{dt} \int \langle y \rangle^{2l} (|u_\beta|^2 + |\theta_\beta|^2 + |h_\beta|^2) dx dy + \int \langle y \rangle^{2l} (\mu |\partial_y u_\beta|^2 + \kappa |\partial_y \theta_\beta|^2 + \nu |\partial_y h_\beta|^2) dx dy \\ & \leq C \delta_0^{-4} (1 + \|(U, \Theta, H)(t)\|_{H^2(\mathbb{T}_x)}^{10}) + C \delta_0^{-4} (\|(u, \theta, h)(t)\|_{\mathcal{H}_l^m}^{12} + \|(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^4) \\ & \quad + \int_\Omega R_1^\beta \cdot \langle y \rangle^{2l} u_\beta dx dy + \int_\Omega R_2^\beta \cdot \langle y \rangle^{2l} \theta^\beta dx dy + \int_\Omega R_3^\beta \cdot \langle y \rangle^{2l} h_\beta dx dy. \end{aligned} \quad (3.126)$$

We claim the following estimate:

$$\begin{aligned} & \int_\Omega R_1^\beta \cdot \langle y \rangle^{2l} u_\beta dx dy + \int_\Omega R_2^\beta \cdot \langle y \rangle^{2l} \theta^\beta dx dy + \int_\Omega R_3^\beta \cdot \langle y \rangle^{2l} h_\beta dx dy \\ & \leq \frac{\mu}{2} \|\partial_y u_\beta\|_{L_t^2(\Omega)}^2 + \frac{\kappa}{2} \|\partial_y \theta_\beta\|_{L_t^2(\Omega)}^2 + \frac{\nu}{2} \|\partial_y h_\beta\|_{L_t^2(\Omega)}^2 + \|\partial_\tau^\beta r_1\|_{L_t^2(\Omega)}^2 \\ & \quad + \|\eta_1 \partial_\tau^\beta r_4\|_{L_t^2(\Omega)}^2 + \|\partial_\tau^\beta r_2\|_{L_t^2(\Omega)}^2 + \|\eta_2 \partial_\tau^\beta r_4\|_{L_t^2(\Omega)}^2 + \|\partial_\tau^\beta r_3\|_{L_t^2(\Omega)}^2 \\ & \quad + \|\eta_3 \partial_\tau^\beta r_4\|_{L_t^2(\Omega)}^2 + C \delta_0^{-4} (1 + \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta (U, \Theta, H)(t)\|_{L^2(\mathbb{T}_x)}^8) \\ & \quad + C \delta_0^{-4} (\|(u, \theta, h)(t)\|_{\mathcal{H}_l^m}^8 + \|(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^4). \end{aligned} \quad (3.127)$$

The combination of (3.126) and (3.127) yields the estimate (3.106) directly. Therefore, we complete the proof of Lemma 3.4. \square

Proof of (3.127). Firstly, we give the estimate for the term $\int_{\Omega} R_1^{\beta} \cdot \langle y \rangle^{2l} u_{\beta} dx dy$. By virtue of Hölder inequality, we find

$$\int_{\Omega} \partial_{\tau}^{\beta} r_1 \cdot \langle y \rangle^{2l} u_{\beta} dx dy \leq \|\partial_{\tau}^{\beta} r_1\|_{L_t^2(\Omega)} \|u_{\beta}\|_{L_t^2(\Omega)} \leq C \|\partial_{\tau}^{\beta} r_1\|_{L_t^2(\Omega)}^2 + C \|u_{\beta}\|_{L_t^2(\Omega)}^2. \quad (3.128)$$

In view of the definition of function (2.2), one attains that

$$\|\partial_{\tau}^{\beta}(U_x \phi' u)\|_{L_t^2(\Omega)} \leq \sum_{\tilde{\beta} \leq \beta} C_{\tilde{\beta}}^{\beta} \|\partial_{\tau}^{\tilde{\beta}}(U_x \phi')\|_{L^{\infty}(\Omega)} \|\partial_{\tau}^{\beta-\tilde{\beta}} u\|_{L_t^2(\Omega)} \leq C \sum_{|\beta| \leq m} \|\partial_{\tau}^{\beta} U_x\|_{L^{\infty}(\mathbb{T}_x)} \|u\|_{\mathcal{H}_t^m}. \quad (3.129)$$

Similarly, it is easy to deduce that

$$\begin{aligned} \|\partial_{\tau}^{\beta}(H_x \phi' h)\|_{L_t^2(\Omega)} &\leq C \sum_{|\beta| \leq m} \|\partial_{\tau}^{\beta} H_x\|_{L^{\infty}(\mathbb{T}_x)} \|h\|_{\mathcal{H}_t^m}, \\ \|\partial_{\tau}^{\beta}(U \phi^{(3)} \theta)\|_{L_t^2(\Omega)} &\leq C \sum_{|\beta| \leq m} \|\partial_{\tau}^{\beta} U\|_{L^{\infty}(\mathbb{T}_x)} \|\theta\|_{\mathcal{H}_t^m}, \\ \|\partial_{\tau}^{\beta}(U \phi'' \theta_y)\|_{L_t^2(\Omega)} &\leq C \sum_{|\beta| \leq m} \|\partial_{\tau}^{\beta} U\|_{L^{\infty}(\mathbb{T}_x)} \|\theta\|_{\mathcal{H}_t^m} + \|U\|_{L^{\infty}(\mathbb{T}_x)} \|\partial_{\tau}^{\beta} \theta_y\|_{L_t^2(\Omega)}. \end{aligned} \quad (3.130)$$

The combination of estimates (3.129) and (3.130) gives that

$$\begin{aligned} \int_{\Omega} \partial_{\tau}^{\beta} I I_1 \cdot \langle y \rangle^{2l} u_{\beta} dx dy &\leq \frac{\kappa}{4} \|\partial_{\tau}^{\beta} \theta_y\|_{L_t^2(\Omega)}^2 + C(1 + \sum_{|\beta| \leq m+2} \|\partial_{\tau}^{\beta}(U, \Theta, H)\|_{L^2(\mathbb{T}_x)}^4) \\ &\quad + C(\|(u, \theta, h)\|_{\mathcal{H}_t^m}^4 + \|u_{\beta}\|_{L_t^2(\Omega)}^2). \end{aligned} \quad (3.131)$$

On the other hand, we get from the inequality (3.96)(or see (B.9)) that

$$\|\partial_{\tau}^{\beta} \theta_y\|_{L_t^2(\Omega)}^2 \leq 2 \|\partial_y \theta_{\beta}\|_{L_t^2(\Omega)}^2 + C \delta_0^{-4} \|h_{\beta}\|_{L_t^2(\Omega)}^2, \quad (3.132)$$

where we have used the estimate (3.99). Then combination of (3.131) and (3.132) yields directly

$$\begin{aligned} \int_{\Omega} \partial_{\tau}^{\beta} I I_1 \cdot \langle y \rangle^{2l} u_{\beta} dx dy &\leq \frac{\kappa}{2} \|\partial_y \theta_{\beta}\|_{L_t^2(\Omega)}^2 + C(1 + \sum_{\beta \leq m+2} \|\partial_{\tau}^{\beta}(U, \Theta, H)\|_{L^2(\mathbb{T}_x)}^4) \\ &\quad + C\|(u, \theta, h)\|_{\mathcal{H}_t^m}^4 + C \delta_0^{-4} \|(u_{\beta}, h_{\beta})\|_{L_t^2(\Omega)}^2. \end{aligned} \quad (3.133)$$

By the definition of communicator operator $[\cdot, \cdot]$, it is easy to deduce that

$$[\partial_{\tau}^{\beta}, (u + U \phi') \partial_x - U_x \phi \partial_y] u = \sum_{0 < \tilde{\beta} \leq \beta} C_{\tilde{\beta}}^{\beta} \partial_{\tau}^{\tilde{\beta}} [(u + U \phi') \partial_x - U_x \phi \partial_y] \partial_{\tau}^{\beta-\tilde{\beta}} u. \quad (3.134)$$

In view of the inequality (A.3), one arrives at directly

$$\begin{aligned} &\|\partial_{\tau}^{\tilde{\beta}} [(u + U \phi') \partial_x - U_x \phi \partial_y] \partial_{\tau}^{\beta-\tilde{\beta}} u\|_{L_t^2(\Omega)} \\ &\leq \|\partial_{\tau}^{\tilde{\beta}} u\|_{\mathcal{H}_t^{m-1}} \|\partial_x u\|_{\mathcal{H}_t^{m-1}} + \|\partial_{\tau}^{\tilde{\beta}} U\|_{L^{\infty}(\mathbb{T}_x)} \|\partial_{\tau}^{\beta-\tilde{\beta}} \partial_x u\|_{L_t^2(\Omega)} \\ &\quad + \|\langle y \rangle^{-1} \partial_{\tau}^{\tilde{\beta}} (U_x \phi)\|_{L^{\infty}(\Omega)} \|\langle y \rangle^{l+1} \partial_{\tau}^{\beta-\tilde{\beta}} \partial_y u\|_{L_t^2(\Omega)} \\ &\leq \|u\|_{\mathcal{H}_t^m}^2 + C \sum_{|\beta| \leq m+1} \|\partial_{\tau}^{\beta} U\|_{L^{\infty}(\mathbb{T}_x)} \|u\|_{\mathcal{H}_t^m}. \end{aligned} \quad (3.135)$$

Then, we can deduce from (3.134) and (3.135) that

$$\|[\partial_\tau^\beta, (u + U\phi')\partial_x - U_x\phi\partial_y]u\|_{L_t^2(\Omega)} \leq \|u\|_{\mathcal{H}_l^m}^2 + C \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta U\|_{L^2(\mathbb{T}_x)} \|u\|_{\mathcal{H}_l^m}. \quad (3.136)$$

Similarly, we can also obtain that

$$\|[\partial_\tau^\beta, (h + H\phi')\partial_x - H_x\phi\partial_y]h\|_{L_t^2(\Omega)} \leq \|h\|_{\mathcal{H}_l^m}^2 + C \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta H\|_{L^2(\mathbb{T}_x)} \|h\|_{\mathcal{H}_l^m}. \quad (3.137)$$

By virtue of the divergence free condition of velocity (i.e. $\partial_x u + \partial_y v = 0$), one arrives at

$$\begin{aligned} \|[\partial_\tau^\beta, U\phi^{(2)}]v\|_{L_t^2(\Omega)} &\leq C \sum_{0 < \tilde{\beta} \leq \beta} \|\partial_\tau^{\tilde{\beta}}(U\phi^{(2)})\|_{L^\infty(\Omega)} \|\langle y \rangle^{-1} \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{-1} u_x\|_{L^2(\Omega)} \\ &\leq C \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta U\|_{L^2(\mathbb{T}_x)} \|u\|_{\mathcal{H}_l^m}. \end{aligned} \quad (3.138)$$

Similarly, we obtain directly

$$\|[\partial_\tau^\beta, H\phi^{(2)}]g\|_{L_t^2(\Omega)} \leq C \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta H\|_{L^2(\mathbb{T}_x)} \|h\|_{\mathcal{H}_l^m}. \quad (3.139)$$

Integrating by part and applying the homogeneous boundary condition (3.103)₄ yields that

$$\begin{aligned} &\mu \int_{\Omega} \partial_y([\partial_\tau^\beta, (\theta + \Theta\phi' + 1)\partial_y]u) \cdot \langle y \rangle^{2l} u_\beta dx dy \\ &= -\mu \int_{\Omega} [\partial_\tau^\beta, (\theta + \Theta\phi' + 1)\partial_y]u \cdot \langle y \rangle^{2l} \partial_y u_\beta dx dy \\ &\quad - 2\mu l \int_{\Omega} [\partial_\tau^\beta, (\theta + \Theta\phi' + 1)\partial_y]u \cdot \langle y \rangle^{2l-1} u_\beta dx dy. \end{aligned}$$

In view of the definition of communicator operator $[\cdot, \cdot]$, it is easy to check that

$$[\partial_\tau^\beta, (\theta + \Theta\phi' + 1)\partial_y]u = \sum_{0 < \tilde{\beta} \leq \beta} C_\beta^\beta \partial_\tau^{\tilde{\beta}-e_i} \partial_\tau^{e_i} \theta \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y u + \sum_{0 < \tilde{\beta} \leq \beta} C_\beta^\beta \partial_\tau^{\tilde{\beta}}(\Theta\phi') \partial_\tau^{\beta-\tilde{\beta}} \partial_y u.$$

Then, we apply the inequality (A.3) and Cauchy inequality to get that

$$\begin{aligned} &\left| \mu \int_{\Omega} \partial_y([\partial_\tau^\beta, (\theta + \Theta\phi' + 1)\partial_y]u) \cdot \langle y \rangle^{2l} u_\beta dx dy \right| \\ &\leq \mu (\|\theta\|_{\mathcal{H}_l^m} + \sum_{|\beta| \leq m} \|\partial_\tau^\beta \Theta\|_{L^\infty(\mathbb{T}_x)}) \|u\|_{\mathcal{H}_l^m} \|\partial_y u_\beta\|_{L_t^2(\Omega)} \\ &\quad + \mu (\|\theta\|_{\mathcal{H}_l^m} + \sum_{|\beta| \leq m} \|\partial_\tau^\beta \Theta\|_{L^\infty(\mathbb{T}_x)}) \|u\|_{\mathcal{H}_l^m} \|u_\beta\|_{L_t^2(\Omega)} \\ &\leq \frac{\mu}{2} \|\partial_y u_\beta\|_{L_t^2(\Omega)}^2 + C(1 + \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta \Theta\|_{L^2(\mathbb{T}_x)}^4) + C(\|(u, \theta)\|_{\mathcal{H}_l^m}^4 + \|u_\beta\|_{L_t^2(\Omega)}^2). \end{aligned} \quad (3.140)$$

By virtue of the divergence free condition of velocity, it is easy to deduce that for $0 < \tilde{\beta} < \beta$

$$\|\partial_\tau^\beta v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y u\|_{L_t^2(\Omega)} = \|\partial_\tau^{\tilde{\beta}-e_i} \partial_y^{-1} \partial_\tau^{e_i+e_2} u \cdot \partial_\tau^{\beta-\tilde{\beta}-e_j} \partial_\tau^{e_j+E_3} u\|_{L_t^2(\Omega)} \leq C \|u\|_{\mathcal{H}_l^m}^2,$$

which implies that

$$\left\| \sum_{0 < \tilde{\beta} < \beta} C_{\tilde{\beta}}^{\beta} \partial_{\tau}^{\beta} v \cdot \partial_{\tau}^{\beta-\tilde{\beta}} \partial_y u \right\|_{L_t^2(\Omega)} \leq C \|u\|_{\mathcal{H}_l^m}^2. \quad (3.141)$$

Similarly, we can find that

$$\left\| \sum_{0 < \tilde{\beta} < \beta} C_{\tilde{\beta}}^{\beta} \partial_{\tau}^{\beta} g \cdot \partial_{\tau}^{\beta-\tilde{\beta}} \partial_y h \right\|_{L_t^2(\Omega)} \leq C \|h\|_{\mathcal{H}_l^m}^2. \quad (3.142)$$

By virtue of the definition of R_u^{β} (see (3.86)), estimates (3.136)-(3.142) and Cauchy inequality, we find

$$\begin{aligned} \int_{\Omega} R_u^{\beta} \cdot \langle y \rangle^{2l} u_{\beta} dx dy &\leq \frac{\mu}{2} \|\partial_y u_{\beta}\|_{L_t^2(\Omega)}^2 + C(1 + \sum_{|\beta| \leq m+2} \|\partial_{\tau}^{\beta}(U, \Theta, H)\|_{L^2(\mathbb{T}_x)}^4) \\ &\quad + C(\|(u, \theta, h)\|_{\mathcal{H}_l^m}^4 + \|u_{\beta}\|_{L_t^2(\Omega)}^2). \end{aligned} \quad (3.143)$$

In view of the definition of r_4 (see (3.91)) estimate (3.105) and Cauchy inequality, it is easy to check that

$$\left| \int_{\Omega} \eta_1 \partial_{\tau}^{\beta} r_4 \cdot \langle y \rangle^{2l} u_{\beta} dx dy \right| \leq \|\eta_1 \partial_{\tau}^{\beta} r_4\|_{L_t^2(\Omega)} \|u_{\beta}\|_{L_t^2(\Omega)} \leq \|\eta_1 \partial_{\tau}^{\beta} r_4\|_{L_t^2(\Omega)}^2 + \|u_{\beta}\|_{L_t^2(\Omega)}^2. \quad (3.144)$$

Similarly, we can find directly

$$\begin{aligned} \|\eta_1 \partial_{\tau}^{\beta}(H_x \phi u)\|_{L_t^2(\Omega)}^2 &\leq C \sum_{\tilde{\beta} \leq \beta} \|\langle y \rangle \eta_1\|_{L^{\infty}(\Omega)}^2 \|\langle y \rangle^{-1} \partial_{\tau}^{\tilde{\beta}} H_x \phi\|_{L^{\infty}(\Omega)}^2 \|\partial_{\tau}^{\beta-\tilde{\beta}} u\|_{L_t^2(\Omega)}^2 \\ &\leq C \delta_0^{-2} (1 + \sum_{|\beta| \leq m+1} \|\partial_{\tau}^{\beta}(U, \Theta, H)\|_{L^2(\mathbb{T}_x)}^8) + C \delta_0^{-2} \|(u, \theta, h)\|_{\mathcal{H}_l^m}^8. \end{aligned} \quad (3.145)$$

With the help of estimates (3.105), (A.7) and divergence free condition of velocity, one arrives at

$$\begin{aligned} \|\eta_1 [\partial_{\tau}^{\beta}, H \phi'] v\|_{L_t^2(\Omega)}^2 &\leq C \sum_{0 < \tilde{\beta} \leq \beta} \|\langle y \rangle^{l+1} \eta_1\|_{L^{\infty}(\Omega)}^2 \|\partial_{\tau}^{\tilde{\beta}}(H \phi')\|_{L^{\infty}(\Omega)}^2 \|\langle y \rangle^{-1} \partial_{\tau}^{\beta-\tilde{\beta}} \partial_y^{-1} u_x\|_{L^2(\Omega)}^2 \\ &\leq C \delta_0^{-2} (1 + \sum_{|\beta| \leq m+1} \|\partial_{\tau}^{\beta}(U, \Theta, H)\|_{L^2(\mathbb{T}_x)}^8) + C \delta_0^{-2} \|(u, \theta, h)\|_{\mathcal{H}_l^m}^8. \end{aligned} \quad (3.146)$$

Obviously, it is easy to check that

$$\begin{aligned} &\eta_1 [\partial_{\tau}^{\beta}, (u + U \phi') \partial_x - U_x \phi \partial_y] \psi \\ &= - \sum_{0 < \tilde{\beta} \leq \beta} C_{\tilde{\beta}}^{\beta} \eta_1 \partial_{\tau}^{\tilde{\beta}} u \cdot \partial_{\tau}^{\beta-\tilde{\beta}} \partial_y^{-1} h_x - \sum_{0 < \tilde{\beta} \leq \beta} C_{\tilde{\beta}}^{\beta} \eta_1 \partial_{\tau}^{\tilde{\beta}}(U \phi') \partial_{\tau}^{\beta-\tilde{\beta}} \partial_y^{-1} h_x \\ &\quad + \sum_{0 < \tilde{\beta} \leq \beta} C_{\tilde{\beta}}^{\beta} \eta_1 \partial_{\tau}^{\tilde{\beta}}(U_x \phi) \partial_{\tau}^{\beta-\tilde{\beta}} h. \end{aligned}$$

In view of the estimate (3.105) and inequality (A.8), one finds

$$\begin{aligned} \|\eta_1 \partial_{\tau}^{\tilde{\beta}} u \partial_{\tau}^{\beta-\tilde{\beta}} \partial_y^{-1} h_x\|_{L_t^2(\Omega)} &\leq \|\langle y \rangle \eta_1\|_{L^{\infty}} \|\partial_{\tau}^{\tilde{\beta}-e_i} \partial_{\tau}^{e_i} u \partial_{\tau}^{\beta-\tilde{\beta}} \partial_y^{-1} h_x\|_{L_{l-1}^2(\Omega)} \\ &\leq \delta_0^{-1} (\|(U, \Theta, H)\|_{L^{\infty}(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_0^3}) \|(u, h)\|_{\mathcal{H}_l^m}^2 \end{aligned} \quad (3.147)$$

Similarly, we obtain that

$$\begin{aligned} &\|\eta_1 \partial_{\tau}^{\tilde{\beta}}(U \phi') \partial_{\tau}^{\beta-\tilde{\beta}} \partial_y^{-1} h_x\|_{L_t^2(\Omega)} \\ &\leq C \delta_0^{-1} (\|(U, \Theta, H)\|_{L^{\infty}(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_l^3}) \|\partial_{\tau}^{\tilde{\beta}} U\|_{L^{\infty}(\mathbb{T}_x)} \|\partial_{\tau}^{\beta-\tilde{\beta}} h_x\|_{L^2(\Omega)}, \end{aligned} \quad (3.148)$$

and

$$\begin{aligned} & \|\eta_1 \partial_\tau^{\tilde{\beta}} (U_x \phi) \partial_\tau^{\beta-\tilde{\beta}} h\|_{L_t^2(\Omega)} \\ & \leq C \delta_0^{-1} (\|(U, \Theta, H)\|_{L^\infty(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_l^3}) \|\partial_\tau^{\tilde{\beta}} U_x\|_{L^\infty(\mathbb{T}_x)} \|\partial_\tau^{\beta-\tilde{\beta}} h\|_{L^2(\Omega)}. \end{aligned} \quad (3.149)$$

The combination of (3.147)-(3.149) yields directly

$$\|\eta_1 [\partial_\tau^\beta, (u + U\phi') \partial_x - U_x \phi \partial_y] \psi\|_{L_t^2(\Omega)} \leq C \delta_0^{-1} (1 + \|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)}^4) + C \delta_0^{-1} \|(u, \theta, h)\|_{\mathcal{H}_l^m}^4. \quad (3.150)$$

By virtue of the divergence free condition of velocity, it is easy to check that

$$\sum_{0 < \tilde{\beta} < \beta} \partial_\tau^{\tilde{\beta}} v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y \psi = - \sum_{0 < \tilde{\beta} < \beta} \partial_\tau^{\tilde{\beta}-e_i} \partial_y^{-1} (\partial_\tau^{e_i} u_x) \partial_\tau^{\beta-\tilde{\beta}-e_i} (\partial_\tau^{e_i} h).$$

Then, the application of (3.105) and (A.7) yields directly

$$\begin{aligned} & \|\eta_1 \partial_\tau^{\tilde{\beta}} v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y \psi\|_{L_t^2(\Omega)} \\ & \leq \|\langle y \rangle^{l+1} \eta_1\|_{L^\infty(\Omega)} \|\partial_\tau^{\tilde{\beta}-e_i} \partial_y^{-1} (\partial_\tau^{e_i} u_x) \partial_\tau^{\beta-\tilde{\beta}-e_i} (\partial_\tau^{e_i} h)\|_{L_{-1}^2(\Omega)} \\ & \leq C \delta_0^{-1} (\|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_l^3}) \|(u, h)\|_{\mathcal{H}_0^m}^2, \end{aligned}$$

which implies that

$$\left\| \sum_{0 < \tilde{\beta} < \beta} \eta_1 \partial_\tau^{\tilde{\beta}} v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y \psi \right\|_{L_t^2(\Omega)} \leq C \delta_0^{-1} (\|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)} + \|(u, \theta, h)\|_{\mathcal{H}_l^3}) \|(u, h)\|_{\mathcal{H}_0^m}^2. \quad (3.151)$$

Hence, the combination of (3.145), (3.146), (3.150) and (3.151) gives that

$$\begin{aligned} & \left| \int_\Omega \eta_1 R_4^\beta \cdot \langle y \rangle^{2l} u_\beta dx dy \right| \\ & \leq C \delta_0^{-2} (1 + \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta (U, \Theta, H)\|_{L^2(\mathbb{T}_x)}^8) + C \delta_0^{-2} (\|(u, \theta, h)\|_{\mathcal{H}_l^m}^8 + \|u_\beta\|_{L_t^2(\Omega)}^2). \end{aligned} \quad (3.152)$$

One applies the estimate (3.105) and Sobolev inequality to get directly

$$\begin{aligned} & \left| \nu \int_\Omega \eta_1 \eta_3 \partial_\tau^\beta h \cdot \langle y \rangle^{2l} u_\beta dx dy \right| \\ & \leq C \|\eta_1\|_{L^\infty(\Omega)} \|\eta_3\|_{L^\infty(\Omega)} \|\partial_\tau^\beta h\|_{L_t^2(\Omega)} \|u_\beta\|_{L_t^2(\Omega)} \\ & \leq C \delta_0^{-2} (1 + \|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)}^4) + C \delta_0^{-2} (\|(u, \theta, h)\|_{\mathcal{H}_l^m}^4 + \|u_\beta\|_{L_t^2(\Omega)}^4). \end{aligned} \quad (3.153)$$

The application of (3.105), (A.7), Hölder inequality and divergence free condition of magnetic field yields

$$\begin{aligned} & \left| \int_\Omega \eta_3 (g - H_x \phi) \partial_\tau^\beta h \cdot \langle y \rangle^{2l} u_\beta dx dy \right| \\ & \leq \|\langle y \rangle^{l+1} \eta_3\|_{L^\infty(\Omega)} (\|\langle y \rangle^{-1} \partial_y^{-1} h_x\|_{L^\infty(\Omega)} + \|\langle y \rangle^{-1} H_x \phi\|_{L^\infty(\Omega)}) \|\partial_\tau^\beta h\|_{L^2(\Omega)} \|u_\beta\|_{L^2(\Omega)} \\ & \leq C \delta_0^{-1} (1 + \|(U, \Theta, H)\|_{H^1(\mathbb{T}_x)}^8) + C \delta_0^{-1} (\|(u, \theta, h)\|_{\mathcal{H}_l^m}^8 + \|u_\beta\|_{L_t^2(\Omega)}^4). \end{aligned} \quad (3.154)$$

By virtue of Hölder inequality, (A.7) and estimate (3.105)₃, we get that

$$\begin{aligned} & \left| \int_\Omega \zeta_1 (\partial_\tau^\beta \psi) \cdot \langle y \rangle^{2l} u_\beta dx dy \right| \\ & \leq \|\langle y \rangle^{l+1} \zeta_1\|_{L^\infty(\Omega)} \|\langle y \rangle^{-1} \partial_\tau^\beta \partial_y^{-1} h\|_{L^2(\Omega)} \|u_\beta\|_{L_t^2(\Omega)} \\ & \leq C \delta_0^{-3} (1 + \sum_{|\beta| \leq 1} \|\partial_\tau^\beta (U, \Theta, H)\|_{H^1(\mathbb{T}_x)}^6) + C \delta_0^{-3} (\|(u, \theta, h)\|_{\mathcal{H}_l^m}^6 + \|u_\beta\|_{L_t^2(\Omega)}^4). \end{aligned} \quad (3.155)$$

Then, the combination of (3.128), (3.133), (3.143), (3.144) and (3.153)-(3.155) yields directly

$$\begin{aligned} \left| \int_{\Omega} R_1^{\beta} \cdot \langle y \rangle^{2l} u_{\beta} dx dy \right| &\leq \frac{\mu}{4} \|\partial_y u_{\beta}\|_{L_t^2(\Omega)}^2 + \frac{\kappa}{2} \|\partial_y \theta_{\beta}\|_{L_t^2(\Omega)}^2 + \|\partial_{\tau}^{\beta} r_1\|_{L_t^2(\Omega)}^2 \\ &\quad + C\delta_0^{-4} (\|(u, \theta, h)\|_{\mathcal{H}_t^m}^8 + \|(u_{\beta}, h_{\beta})\|_{L_t^2(\Omega)}^4) \\ &\quad + C\delta_0^{-4} (1 + \sum_{|\beta| \leq m+2} \|\partial_{\tau}^{\beta}(U, \Theta, H)\|_{L^2(\mathbb{T}_x)}^8). \end{aligned} \quad (3.156)$$

Similarly, we can also obtain that

$$\begin{aligned} \int_{\Omega} R_2^{\beta} \cdot \langle y \rangle^{2l} \theta_{\beta} dx dy &\leq \frac{\mu}{4} \|\partial_y u_{\beta}\|_{L_t^2(\Omega)}^2 + \frac{\nu}{2} \|\partial_y h_{\beta}\|_{L_t^2(\Omega)}^2 + \|\partial_{\tau}^{\beta} r_2\|_{L_t^2(\Omega)}^2 \\ &\quad + \|\eta_2 \partial_{\tau}^{\beta} r_4\|_{L_t^2(\Omega)}^2 + C\delta_0^{-4} (\|(u, \theta, h)\|_{\mathcal{H}_t^m}^8 + \|(\theta_{\beta}, h_{\beta})\|_{L_t^2(\Omega)}^4) \\ &\quad + C\delta_0^{-4} (1 + \sum_{|\beta| \leq m+2} \|\partial_{\tau}^{\beta}(U, \Theta, H)\|_{L^2(\mathbb{T}_x)}^8), \end{aligned} \quad (3.157)$$

and

$$\begin{aligned} \int_{\Omega} R_3^{\beta} \cdot \langle y \rangle^{2l} h_{\beta} dx dy &\leq \|\partial_{\tau}^{\beta} r_3\|_{L_t^2(\Omega)}^2 + \|\eta_3 \partial_{\tau}^{\beta} r_4\|_{L_t^2(\Omega)}^2 + C\delta_0^{-3} \|(u, \theta, h)\|_{\mathcal{H}_t^m}^6 \\ &\quad + C\delta_0^{-3} \|h_{\beta}\|_{L_t^2(\Omega)}^6 + C\delta_0^{-3} (1 + \sum_{|\beta| \leq m+2} \|\partial_{\tau}^{\beta}(U, \Theta, H)\|_{L^2(\mathbb{T}_x)}^6). \end{aligned} \quad (3.158)$$

Therefore, the combination of (3.156)-(3.158) completes the proof of (3.127). \square

3.4. Closeness of the a priori estimates

In this subsection, we will give the proof for the Proposition 3.1 by collecting all the estimates obtained in this section. Indeed, the combination of estimates (3.95), (3.96) and (3.99) yields immediately

$$\begin{aligned} \|(u, \theta, h)(t)\|_{\mathcal{H}_t^m}^2 &= \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^{\alpha}(u, \theta, h)(t)\|_{L_{t+k}^2(\Omega)}^2 + \sum_{|\beta|=m} \|\partial_{\tau}^{\beta}(u, \theta, h)(t)\|_{L_t^2(\Omega)}^2 \\ &\leq \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^{\alpha}(u, \theta, h)(t)\|_{L_{t+k}^2(\Omega)}^2 + 36\delta_0^{-4} \sum_{|\beta|=m} \|(u_{\beta}, \theta_{\beta}, h_{\beta})(t)\|_{L_t^2(\Omega)}^2, \end{aligned} \quad (3.159)$$

and

$$\begin{aligned} \|\partial_y(u, \theta, h)(t)\|_{\mathcal{H}_t^m}^2 &= \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^{\alpha} \partial_y(u, \theta, h)(t)\|_{L_{t+k}^2(\Omega)}^2 + \sum_{|\beta|=m} \|\partial_{\tau}^{\beta} \partial_y(u, \theta, h)(t)\|_{L_t^2(\Omega)}^2 \\ &\leq \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^{\alpha} \partial_y(u, \theta, h)(t)\|_{L_{t+k}^2(\Omega)}^2 + 2 \sum_{|\beta|=m} \|\partial_y(u_{\beta}, \theta_{\beta}, h_{\beta})(t)\|_{L_t^2(\Omega)}^2 \\ &\quad + 72\delta_0^{-4} \sum_{|\beta|=m} \|h_{\beta}(t)\|_{L_t^2(\Omega)}^2. \end{aligned} \quad (3.160)$$

Then, we can obtain the following proposition that will play an important role in giving the proof for the Proposition 3.1.

Proposition 3.5. *Under the assumptions of Proposition 3.1, there exists a constant $C > 0$, depending only on m, M_0 and ϕ , such that*

$$\sup_{0 \leq s \leq t} \|(u, \theta, h)(s)\|_{\mathcal{H}_t^m} \leq \left\{ F(0) + \int_0^t G(\tau) d\tau \right\}^{\frac{1}{2}} \left\{ 1 - C\delta_0^{-8} t [F(0) + \int_0^t G(\tau) d\tau]^5 \right\}^{-\frac{1}{10}}. \quad (3.161)$$

for small time t , where the quantities $F(0)$ and $G(t)$ are defined as follows

$$F(0) \triangleq \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(u, \theta, h)(0)\|_{L_{l+k}^2(\Omega)}^2 + 36\delta_0^{-4} \sum_{|\beta|=m} \|(u_{\beta 0}, \theta_{\beta 0}, h_{\beta 0})\|_{L_l^2(\Omega)}^2, \quad (3.162)$$

$$\begin{aligned} G(t) \triangleq & C\delta_0^{-8} \left(\sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, \Theta, H, P)(t)\|_{L^2(\mathbb{T}_x)}^2 + 1 \right)^5 + C \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1, r_2, r_3)(t)\|_{L_{l+k}^2(\Omega)}^2 \\ & + C\delta_0^{-4} \sum_{|\beta|=m} \left\{ \|\partial_\tau^\beta(r_1, r_2, r_3)(t)\|_{L_l^2(\Omega)}^2 + 4\delta_0^{-4} \|\partial_\tau^\beta r_4(t)\|_{L_{-1}^2(\Omega)}^2 \right\}. \end{aligned} \quad (3.163)$$

Also, we have that for $i = 1, 2$,

$$\begin{aligned} & \|\langle y \rangle^{l+1} \partial_y^i(u, \theta, h)(t, x, y)\|_{L^\infty(\Omega)} \\ & \leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, \theta_0, h_0)(x, y)\|_{L^\infty(\Omega)} \\ & \quad + Ct \left\{ F(0) + \int_0^t G(\tau) d\tau \right\}^{\frac{1}{2}} \left\{ 1 - C\delta_0^{-8} t [F(0) + \int_0^t G(\tau) d\tau]^5 \right\}^{-\frac{1}{10}}, \end{aligned} \quad (3.164)$$

and

$$h(t, x, y) \geq h_0(x, y) - Ct \left\{ F(0) + \int_0^t G(\tau) d\tau \right\}^{\frac{1}{2}} \left\{ 1 - C\delta_0^{-8} t [F(0) + \int_0^t G(\tau) d\tau]^5 \right\}^{-\frac{1}{10}}. \quad (3.165)$$

Proof. Indeed, multiplying (3.106) by $36\delta_0^{-4}$ and adding with (3.10), then we find

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(u, \theta, h)(t)\|_{L_{l+k}^2(\Omega)}^2 + 36\delta_0^{-4} \sum_{|\beta|=m} \|(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_l^2(\Omega)}^2 \right) \\ & + \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha \partial_y(u, \theta, h)(t)\|_{L_{l+k}^2(\Omega)}^2 + 36\delta_0^{-4} \sum_{|\beta|=m} \|\partial_y(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_l^2(\Omega)}^2 \\ & \leq \delta_1 C \|\partial_y(u, \theta, h)(t)\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1} (\|(u, \theta, h)(t)\|_{\mathcal{H}_l^m}^8 + 1) + \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1, r_2, r_3)(t)\|_{L_{l+k}^2(\Omega)}^2 \\ & + 36\delta_0^{-4} \sum_{|\beta|=m} \left\{ \sum_{i=1}^3 \|\partial_\tau^\beta r_i\|_{L_l^2(\Omega)}^2 + \|\eta_i \partial_\tau^\beta r_4\|_{L_l^2(\Omega)}^2 \right\} + C\delta_0^{-4} \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, \Theta, H)(t)\|_{L^2(\mathbb{T}_x)}^{10} \\ & + C\delta_0^{-8} (\|(u, \theta, h)(t)\|_{\mathcal{H}_l^m}^{12} + \|(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_l^2(\Omega)}^4 + 1). \end{aligned} \quad (3.166)$$

Choosing δ_1 small enough in (3.166) and applying the estimates (3.95), (3.96), (3.159), and (3.160), it is easy to deduce that

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(u, \theta, h)(t)\|_{L_{l+k}^2(\Omega)}^2 + 36\delta_0^{-4} \sum_{|\beta|=m} \|(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_l^2(\Omega)}^2 \right) \\ & + c_1 \left(\sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha \partial_y(u, \theta, h)(t)\|_{L_{l+k}^2(\Omega)}^2 + 36\delta_0^{-4} \sum_{|\beta|=m} \|\partial_y(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_l^2(\Omega)}^2 \right) \\ & \leq C\delta_0^{-8} \left(\sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(u, \theta, h)(t)\|_{L_{l+k}^2(\Omega)}^2 + 36\delta_0^{-4} \sum_{|\beta|=m} \|(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_l^2(\Omega)}^2 \right) \\ & + C \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1, r_2, r_3)(t)\|_{L_{l+k}^2(\Omega)}^2 + C\delta_0^{-8} \left(\sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, \Theta, H)(t)\|_{L^2(\mathbb{T}_x)}^2 + 1 \right)^5 \\ & + C\delta_0^{-4} \sum_{|\beta|=m} \left\{ \|\partial_\tau^\beta(r_1, r_2, r_3)(t)\|_{L_l^2(\Omega)}^2 + 4\delta_0^{-4} \|\partial_\tau^\beta r_4(t)\|_{L_{-1}^2(\Omega)}^2 \right\}. \end{aligned} \quad (3.167)$$

Then, we apply the comparison principle of ordinary differential equation to (3.167) get that

$$\begin{aligned} & \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(u, \theta, h)(t)\|_{L_{l+k}^2(\Omega)}^2 + 36\delta_0^{-4} \sum_{|\beta|=m} \|(u_\beta, \theta_\beta, h_\beta)(t)\|_{L_l^2(\Omega)}^2 \\ & + c_1 \int_0^t \left(\sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha \partial_y(u, \theta, h)(\tau)\|_{L_{l+k}^2(\Omega)}^2 + 36\delta_0^{-4} \sum_{|\beta|=m} \|\partial_y(u_\beta, \theta_\beta, h_\beta)(\tau)\|_{L_l^2(\Omega)}^2 \right) d\tau \\ & \leq [F(0) + \int_0^t G(\tau) d\tau] \left\{ 1 - C\delta_0^{-8} t [F(0) + \int_0^t G(\tau) d\tau]^5 \right\}^{-\frac{1}{5}}, \end{aligned}$$

which, together with (3.159), yields directly

$$\|(u, \theta, h)(t)\|_{\mathcal{H}_l^m} \leq \left\{ F(0) + \int_0^t G(\tau) d\tau \right\}^{\frac{1}{2}} \left\{ 1 - C\delta_0^{-8} t [F(0) + \int_0^t G(\tau) d\tau]^5 \right\}^{-\frac{1}{10}}.$$

As we know, we have for $i = 1, 2$,

$$\langle y \rangle^{l+1} \partial_y^i(u, \theta, h)(t, x, y) = \langle y \rangle^{l+1} \partial_y^i(u_0, \theta_0, h_0)(x, y) + \int_0^t \langle y \rangle^{l+1} \partial_y^i \partial_s(u, \theta, h)(s, x, y) ds, \quad (3.168)$$

and

$$h(t, x, y) = h_0(x, y) + \int_0^t \partial_s h(s, x, y) ds.$$

In view of the Sobolev embedding theorem and the relation (3.168), one arrives at

$$\begin{aligned} & \|\langle y \rangle^{l+1} \partial_y^i(u, \theta, h)(t, x, y)\|_{L^\infty(\Omega)} \\ & \leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, \theta_0, h_0)(x, y)\|_{L^\infty(\Omega)} + Ct \sup_{0 \leq s \leq t} \|(u, \theta, h)(s)\|_{\mathcal{H}_l^5} \\ & \leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, \theta_0, h_0)(x, y)\|_{L^\infty(\Omega)} \\ & \quad + Ct \left\{ F(0) + \int_0^t G(\tau) d\tau \right\}^{\frac{1}{2}} \left\{ 1 - C\delta_0^{-8} t [F(0) + \int_0^t G(\tau) d\tau]^5 \right\}^{-\frac{1}{10}} \end{aligned}$$

and similarly, we also have

$$\begin{aligned} h(t, x, y) & \geq h_0(x, y) - \int_0^t \|\partial_\tau h(\tau, x, y)\|_{L^\infty(\Omega)} d\tau \\ & \geq h_0(x, y) - Ct \sup_{0 \leq s \leq t} \|h(s)\|_{\mathcal{H}_0^3} \\ & \geq h_0(x, y) - Ct \left\{ F(0) + \int_0^t G(\tau) d\tau \right\}^{\frac{1}{2}} \left\{ 1 - C\delta_0^{-8} t [F(0) + \int_0^t G(\tau) d\tau]^5 \right\}^{-\frac{1}{10}}. \end{aligned}$$

Therefore, we complete the proof of Proposition 3.5. \square

Proof of Proposition 3.1. Indeed, it is easy to deduce from the definition of (3.163) that

$$G(t) \leq C\delta_0^{-8} M_0^{10}. \quad (3.169)$$

On the other hand, it is easy to check that $D^\alpha(u, \theta, h)(0, x, y)$, $|\alpha| \leq m$ can be expressed by the spatial derivatives of initial data (u_0, θ_0, h_0) up to order $2m$. Then, we get that

$$F(0) \leq \delta_0^{-8} \mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_l^{2m}(\Omega)}). \quad (3.170)$$

Substituting the estimates (3.169) and (3.170) into (3.161)-(3.165), then it is easy to get the estimates (3.2)-(3.4). Therefore, we complete the proof of Proposition 3.1. \square

4 Local-in-time Existence and Uniqueness

In this section, we will establish the local-in-time existence and uniqueness of solution to the nonlinear MHD boundary layer problem (2.6).

4.1. Local-in-time Existence for the Boundary Layer Equations

In this subsection, we investigate a parabolic regularized system for the nonlinear problem (2.6), which we can obtain the local-in-time existence of solution by using the classical energy estimates. More precisely, for a small parameter $0 < \epsilon < 1$, one investigates the following system:

$$\left\{ \begin{array}{l} \partial_t u^\epsilon + [(u^\epsilon + U\phi')\partial_x + (v^\epsilon - U_x\phi)\partial_y]u^\epsilon - [(h^\epsilon + H\phi')\partial_x + (g^\epsilon - H_x\phi)\partial_y]h^\epsilon \\ \quad - \mu\partial_y[(\theta^\epsilon + \Theta\phi'(y) + 1)\partial_y u^\epsilon] - \epsilon\partial_x^2 u^\epsilon + U_x\phi'u^\epsilon + U\phi''v^\epsilon - H_x\phi'h^\epsilon \\ \quad - H\phi''g^\epsilon - U\phi^{(3)}\theta^\epsilon - U\phi''\theta_y^\epsilon = r_1^\epsilon, \\ c_v\{\partial_t\theta^\epsilon + (u^\epsilon + U\phi')\partial_x\theta^\epsilon + (v^\epsilon - U_x\phi)\partial_y\theta^\epsilon\} - \kappa\partial_y^2\theta^\epsilon - \epsilon\partial_x^2\theta^\epsilon + c_v\Theta_x\phi'u^\epsilon + c_v\Theta\phi''v^\epsilon \\ \quad - \mu\theta^\epsilon(u_y^\epsilon)^2 - \mu(U\phi'')^2\theta^\epsilon - 2\mu U\phi''\theta^\epsilon\partial_y u^\epsilon - \mu\Theta\phi'(\partial_y u^\epsilon)^2 - 2\mu\Theta U\phi'\phi''\partial_y u^\epsilon \\ \quad - 2\mu U\phi''\partial_y u^\epsilon - \mu(\partial_y u^\epsilon)^2 - \nu(\partial_y h^\epsilon)^2 - 2\nu H\phi'\partial_y h^\epsilon = r_2^\epsilon, \\ \partial_t h^\epsilon + [(u^\epsilon + U\phi')\partial_x + (v^\epsilon - U_x\phi)\partial_y]h^\epsilon - [(h^\epsilon + H\phi')\partial_x + (g^\epsilon - H_x\phi)\partial_y]u^\epsilon \\ \quad - \nu\partial_y^2 h^\epsilon - \epsilon\partial_x^2 h^\epsilon + H_x\phi'u^\epsilon + H\phi''v^\epsilon - U_x\phi'h^\epsilon - U\phi''g^\epsilon = r_3^\epsilon, \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, \quad \partial_x h^\epsilon + \partial_y g^\epsilon = 0, \\ (u^\epsilon, \theta^\epsilon, h^\epsilon)|_{t=0} = (u_0, \theta_0, h_0)(x, y) \quad (u^\epsilon, v^\epsilon, \theta^\epsilon, \partial_y h^\epsilon, g^\epsilon)|_{y=0} = 0, \end{array} \right. \quad (4.1)$$

where the source term $(r_1^\epsilon, r_2^\epsilon, r_3^\epsilon)$ is defined by

$$(r_1^\epsilon, r_2^\epsilon, r_3^\epsilon)(t, x, y) \triangleq (r_1, r_2, r_3)(t, x, y) + \epsilon(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon, \tilde{r}_3^\epsilon)(t, x, y). \quad (4.2)$$

Here, (r_1, r_2, r_3) is source term of the original problem (2.6), and $(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon, \tilde{r}_3^\epsilon)$ is constructed to ensure that the initial data (u_0, θ_0, h_0) also satisfies the compatibility conditions of (4.1) up to the order of m . Indeed, we can use the given functions $\partial_t^i(u, \theta, h)(0, x, y)$, $0 \leq i \leq m$, which can be derived from the equations and initial data of (2.6) by induction with respect to i , and it follows that $\partial_t^i(u, \theta, h)(0, x, y)$ can be expressed as polynomials of the spatial derivatives, up to order $2i$, of the initial data (u_0, θ_0, h_0) . Then, similar to [42], one can choose the corrector $(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon, \tilde{r}_3^\epsilon)$ in the following form:

$$(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon, \tilde{r}_3^\epsilon)(t, x, y) \triangleq - \sum_{i=0}^m \frac{t^i}{i!} \partial_t^i \partial_x^2(u, \theta, h)(0, x, y), \quad (4.3)$$

which, yields that by a direct calculation

$$\partial_t^i(u^\epsilon, \theta^\epsilon, h^\epsilon)(0, x, y) = \partial_t^i(u, \theta, h)(0, x, y), \quad 0 \leq i \leq m. \quad (4.4)$$

Similarly, we can derive that $\psi^\epsilon \triangleq \partial_y^{-1} h^\epsilon$ satisfies

$$\partial_t \psi^\epsilon + [(u^\epsilon + U\phi')\partial_x + (v^\epsilon - U_x\phi)\partial_y]\psi^\epsilon + H_x\phi u^\epsilon - \nu\partial_y^2 \psi^\epsilon - \epsilon\partial_x^2 \psi^\epsilon = r_4^\epsilon, \quad (4.5)$$

where

$$r_4^\epsilon \triangleq r_4 + \epsilon\tilde{r}_4^\epsilon, \quad \tilde{r}_4^\epsilon \triangleq - \sum_{i=0}^m \frac{t^i}{i!} \int_0^y \partial_t^i \partial_x^2 h(0, x, z) dz. \quad (4.6)$$

Then, one attains directly for $\alpha = (\beta, k) = (\beta_1, \beta_2, k)$ with $|\alpha| \leq m$,

$$\|D^\alpha(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon, \tilde{r}_3^\epsilon)\|_{L_{l+k}^2(\Omega)}, \|\partial_\tau^\beta \tilde{r}_4^\epsilon\|_{L_{-1}^2(\Omega)} \leq t^{i-\beta_1} \sum_{\beta_1 \leq i \leq m} \mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_t^{2i+2+\beta_2+k}}). \quad (4.7)$$

Now, we are can obtain the following proposition by the previous estimate.

Proposition 4.1. *Under the hypotheses of Theorem 2.1, there exist a positive time $0 < T_* \leq T$, independent of ϵ , and a solution $(u^\epsilon, v^\epsilon, \theta^\epsilon, h^\epsilon, g^\epsilon)$ to the initial boundary value problem (4.1) with $(u^\epsilon, \theta^\epsilon, h^\epsilon) \in L^\infty(0, T_*; \mathcal{H}_l^m)$, which satisfies the following uniform estimates in ϵ :*

$$\sup_{0 \leq t \leq T_*} \|(u^\epsilon, \theta^\epsilon, h^\epsilon)(t)\|_{\mathcal{H}_l^m} \leq 2F(0)^{\frac{1}{2}}, \quad (4.8)$$

where $F(0)$ is given by (3.162). Moreover, for $(t, x, y) \in [0, T_*] \times \Omega$, it holds on

$$|\langle y \rangle^{l+1} \partial_y^i(u, \theta, h)(t, x, y)| \leq \delta_0^{-1}, \quad i = 1, 2, \quad (4.9)$$

and

$$h(t, x, y) + H(t, x)\phi'(y) \geq \delta_0. \quad (4.10)$$

Proof. First of all, one can establish the a priori estimates as in Proposition 3.5 for the regularized boundary layer systems (4.1). Then, the standard continuity argument helps us obtain the existence of solution in a time interval $[0, T_*]$, $T_* > 0$ independent of ϵ . Hence, the only task for us is to determine the uniform lifespan T_* , and verify estimates (4.8)- (4.10). Indeed, we apply the Proposition 3.5 to get that

$$\sup_{0 \leq s \leq t} \|(u, \theta, h)(s)\|_{\mathcal{H}_l^m} \leq \left\{ F(0) + \int_0^t G^\epsilon(\tau) d\tau \right\}^{\frac{1}{2}} \left\{ 1 - C\delta_0^{-8} t [F(0) + \int_0^t G^\epsilon(\tau) d\tau]^5 \right\}^{-\frac{1}{10}}, \quad (4.11)$$

where the function $G^\epsilon(t)$ is defined as follows:

$$\begin{aligned} G^\epsilon(t) &\triangleq C\delta_0^{-8} \left(1 + \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, \Theta, H, P)(t)\|_{L^2(\mathbb{T}_x)}^2 \right)^5 + C \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1^\epsilon, r_2^\epsilon, r_3^\epsilon)(t)\|_{L_{l+k}^2(\Omega)}^2 \\ &\quad + C\delta_0^{-4} \sum_{|\beta|=m} \left\{ \|\partial_\tau^\beta(r_1^\epsilon, r_2^\epsilon, r_3^\epsilon)\|_{L_l^2(\Omega)}^2 + 4\delta_0^{-4} \|\partial_\tau^\beta r_4^\epsilon\|_{L_{-1}^2(\Omega)}^2 \right\}. \end{aligned} \quad (4.12)$$

Recalling the definition of $G(t)$ (see (3.163)), it is easy to check that

$$\begin{aligned} G^\epsilon(t) &= G(t) + C\epsilon^2\delta_0^{-4} \sum_{|\beta|=m} \left\{ \|\partial_\tau^\beta(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon, \tilde{r}_3^\epsilon)\|_{L_l^2(\Omega)}^2 + 4\delta_0^{-4} \|\partial_\tau^\beta \tilde{r}_4^\epsilon\|_{L_{-1}^2(\Omega)}^2 \right\} \\ &\quad + C\epsilon^2 \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon, \tilde{r}_3^\epsilon)(t)\|_{L_{l+k}^2(\Omega)}^2. \end{aligned}$$

Then, the combination of (3.169) and (4.7) yields immediately

$$G^\epsilon(t) \leq C\delta_0^{-8} M_0^{10} + \epsilon^2\delta_0^{-8} \mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_t^{3m+2}}) \leq \delta_0^{-8} \mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_t^{3m+2}}). \quad (4.13)$$

Therefore, we choose the existence time

$$T_1 \triangleq \min \left\{ \frac{\delta_0^8 F(0)}{\mathcal{P}(M_0 + \|(u_0, \theta_0, h_0)\|_{H_t^{3m+2}})}, \frac{3\delta_0^8}{128CF(0)^5} \right\}$$

in (4.11), we can get the estimate (4.8) for all $T_* \leq T_1$.

On the other hand, by virtue of the Proposition 3.5, it is easy to deduce the following bounds for $\langle y \rangle^{l+1} \partial_y^i(u^\epsilon, \theta^\epsilon, h^\epsilon)$, $i = 1, 2$ that

$$\begin{aligned} & \|\langle y \rangle^{l+1} \partial_y^i(u^\epsilon, \theta^\epsilon, h^\epsilon)(t, x, y)\|_{L^\infty(\Omega)} \\ & \leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, \theta_0, h_0)(x, y)\|_{L^\infty(\Omega)} + Ct \sup_{0 \leq s \leq t} \|(u^\epsilon, \theta^\epsilon, h^\epsilon)(s)\|_{\mathcal{H}_l^5} \\ & \leq (2\delta_0)^{-1} + 2CF(0)^{\frac{1}{2}}t. \end{aligned} \quad (4.14)$$

Then, choosing the existence of time

$$T_2 \triangleq \min \left\{ T_1, \frac{1}{4C\delta_0 F(0)^{\frac{1}{2}}} \right\},$$

in (4.14), one can find the estimate (4.9). Similarly, choosing the existence of time

$$T_3 \triangleq \min \left\{ T_2, \frac{\delta_0}{C^2(M_0 + 2F(0)^{\frac{1}{2}})^2} \right\},$$

then we apply the estimate in Proposition 3.5 to deduce that

$$\begin{aligned} & h^\epsilon(t, x, y) + H(t, x)\phi'(y) \\ & \geq h_0(x, y) + H(t, x)\phi'(y) - Ct \sup_{0 \leq s \leq t} \|h(s)\|_{\mathcal{H}_0^3} \\ & \geq h_0(x, y) + (H(t, x) - H(0, x))\phi'(y) - 2CF(0)^{\frac{1}{2}}t \\ & \geq 2\delta_0 - C(M_0 + 2F(0)^{\frac{1}{2}})t^{\frac{1}{2}} \geq \delta_0, \end{aligned}$$

where we have used the Hölder inequality. Therefore, we find the lifespan $T_* = T_3$ and establish the estimates (4.8) -(4.10), and consequently complete the proof of the Proposition 4.1. \square

Proof of Local Existence. Indeed, we get the local existence of solutions $(u^\epsilon, v^\epsilon, \theta^\epsilon, h^\epsilon, g^\epsilon)$ to the non-linear MHD boundary layer problem (2.6) and their uniform estimates in ϵ . Now, by letting $\epsilon \rightarrow 0$ one obtains the solution to the original problem (2.6) by applying some compactness argument. Indeed, from the uniform estimates (4.8), by the Lions-Aubin lemma and the compact embedding of $H_l^m(\Omega)$ in $H_{loc}^{m'}(\Omega)$ for $m' < m$ (see [7, Lemma 6.2]), we know that there exists

$$(u, \theta, h) \in L^\infty(0, T_*; \mathcal{H}_l^m) \cap (\cap_{m' < m-1} C^1([0, T_*]; H_{loc}^{m'}(\Omega))),$$

such that, up to a subsequences,

$$\begin{aligned} & \partial_t^i(u^\epsilon, \theta^\epsilon, h^\epsilon) \xrightarrow{*} \partial_t^i(u, \theta, h), \quad \text{in } L^\infty(0, T_*; H_l^{m-i}(\Omega)), \quad 0 \leq i \leq m, \\ & (u^\epsilon, \theta^\epsilon, h^\epsilon) \rightarrow (u, \theta, h) \quad \text{in } C^1([0, T_*]; H_{loc}^{m'}(\Omega)). \end{aligned} \quad (4.15)$$

On the other hand, by virtue of $(\partial_x u^\epsilon, \partial_x h^\epsilon) \in Lip(\Omega_{T_*})$, we find the uniform convergence of $(\partial_x u^\epsilon, \partial_x h^\epsilon)$. Then, we can obtain the pointwise convergence for (v^ϵ, g^ϵ) , i.e.,

$$(v^\epsilon, g^\epsilon) = \left(- \int_0^y \partial_x u^\epsilon dz, - \int_0^y \partial_x h^\epsilon dz \right) \rightarrow \left(- \int_0^y \partial_x u dz, - \int_0^y \partial_x h dz \right) \triangleq (v, g). \quad (4.16)$$

Now, we can pass the limit $\epsilon \rightarrow 0$ in problem (4.1), and obtain that (u, v, θ, h, g) , solves the original problem (2.6). By virtue of the definition of function space \mathcal{H}_l^m (see (2.1)), it is easy to get $(u, \theta, h) \in \cap_{i=0}^m W^{i, \infty}(0, T_*; H_l^{m-i}(\Omega))$ from the fact $(u, \theta, h) \in L^\infty(0, T_*; \mathcal{H}_l^m)$, then one proves (2.13) directly. On the other hand, the relation (2.14) follows directly by combining the divergence free conditions $v = -\partial_y^{-1} \partial_x u$, $g = -\partial_y^{-1} \partial_x h$, with (A.7). Therefore, we prove the local-in-time existence of Theorem 2.1. \square

4.2. Uniqueness for the Boundary Layer Equations

In this subsection, we will give the uniqueness of the solution to the nonlinear MHD boundary layer problem (2.6). Let $(u_1, v_1, \theta_1, h_1, g_1)$ and $(u_2, v_2, \theta_2, h_2, g_2)$ be two solutions in the lifespan $[0, T_*]$, constructed in the previous subsection, with respect to the initial data $(u_{10}, \theta_{10}, h_{10})$ and $(u_{20}, \theta_{20}, h_{20})$ respectively. Set

$$(\tilde{u}, \tilde{v}, \tilde{\theta}, \tilde{h}, \tilde{g}) \triangleq (u_1 - u_2, v_1 - v_2, \theta_1 - \theta_2, h_1 - h_2, g_1 - g_2),$$

then we obtain the following systems:

$$\left\{ \begin{array}{l} \partial_t \tilde{u} + [(u_1 + U\phi')\partial_x + (v_1 - U_x\phi)\partial_y]\tilde{u} - [(h_1 + H\phi')\partial_x + (g_1 - H_x\phi)\partial_y]\tilde{h} \\ \quad - \mu\partial_y[(\theta_1 + \Theta\phi' + 1)\partial_y\tilde{u}] + (\partial_x u_2 + U_x\phi')\tilde{u} + (\partial_y u_2 + U\phi'')\tilde{v} - (\partial_x h_2 + H_x\phi')\tilde{h} \\ \quad - (\partial_y h_2 + H\phi'')\tilde{g} - \mu\partial_y(\tilde{\theta}\partial_y u_2) - U\phi^{(3)}\tilde{\theta} - U\phi''\partial_y\tilde{\theta} = 0, \\ c_v \left[\partial_t \tilde{\theta} + (u_1 + U\phi')\partial_x \tilde{\theta} + (v_1 - U_x\phi)\partial_y \tilde{\theta} \right] - \kappa\partial_y^2 \tilde{\theta} + c_v(\partial_x \theta_2 + \partial_x \Theta\phi')\tilde{u} + c_v(\partial_y \theta_2 + \Theta\phi'')\tilde{v} \\ \quad - \mu\tilde{\theta}[(U\phi'')^2 + (\partial_y u_1)^2 + 2U\phi''\partial_y u_1] - \nu\partial_y \tilde{h}(2H\phi' + \partial_y h_1 + \partial_y h_2) - \mu a_1 \partial_y \tilde{u} = 0, \\ \partial_t \tilde{h} + [(u_1 + U\phi')\partial_x + (v_1 - U_x\phi)\partial_y]\tilde{h} - [(h_1 + H\phi')\partial_x + (g_1 - H_x\phi)\partial_y]\tilde{u} - \nu\partial_y^2 \tilde{h} \\ \quad + (\partial_x h_2 + H_x\phi')\tilde{u} + (\partial_y h_2 + H\phi'')\tilde{v} - (\partial_x u_2 + U_x\phi')\tilde{h} - (\partial_y u_2 + U\phi'')\tilde{g} = 0, \\ \partial_x \tilde{u} + \partial_y \tilde{v} = 0, \quad \partial_x \tilde{h} + \partial_y \tilde{g} = 0, \\ (\tilde{u}, \tilde{\theta}, \tilde{h})|_{t=0} = (u_{10} - u_{20}, \theta_{10} - \theta_{20}, h_{10} - h_{20}), \quad (\tilde{u}, \tilde{v}, \tilde{\theta}, \partial_y \tilde{h}, \tilde{g})|_{y=0} = 0, \end{array} \right. \quad (4.17)$$

where the function a_1 is defined by

$$a_1 \triangleq 2\Theta U\phi'\phi'' + 2U\phi'' + \theta_2\partial_y u_1 + \theta_2\partial_y u_2 + 2U\phi''\theta_2 + \Theta\phi'\partial_y u_1 + \Theta\phi'\partial_y u_2 + \partial_y u_1 + \partial_y u_2.$$

On the other hand, denote $\tilde{\psi} = \partial_y^{-1}\tilde{h} = \partial_y^{-1}(h_2 - h_1)$, then it is easy to check that $\tilde{\psi}$ satisfies the following equation:

$$\partial_t \tilde{\psi} + [(u_1 + U\phi')\partial_x + (v_1 - U_x\phi)\partial_y]\tilde{\psi} - \nu\partial_y^2 \tilde{\psi} - (g_2 - H_x\phi)\tilde{u} + (h_2 + H\phi')\tilde{v} = 0. \quad (4.18)$$

Let us introduce the following new quantities:

$$\bar{u} \triangleq \tilde{u} - \eta_4 \tilde{\psi}, \quad \bar{\theta} \triangleq \tilde{\theta} - \eta_5 \tilde{\psi}, \quad \bar{h} \triangleq \tilde{h} - \eta_6 \tilde{\psi}, \quad (4.19)$$

where

$$\eta_4 = \frac{\partial_y u_2 + U\phi''}{h_2 + H\phi'}, \quad \eta_5 = \frac{\partial_y \theta_2 + \Theta\phi''}{h_2 + H\phi'}, \quad \eta_6 = \frac{\partial_y h_2 + H\phi''}{h_2 + H\phi'}.$$

By virtue of the equations (4.17), (4.18) and the definition (4.19), it is easy to verify that $(\bar{u}, \bar{\theta}, \bar{h})$ admits the following problem:

$$\left\{ \begin{array}{l} \partial_t \bar{u} + [(u_1 + U\phi')\partial_x + (v_1 - U_x\phi)\partial_y]\bar{u} - \mu\partial_y[(\theta_1 + \Theta\phi' + 1)\partial_y\bar{u}] - \mu\partial_y[(\theta_1 + \Theta\phi' + 1)\partial_y\eta_4\tilde{\psi}] \\ \quad - \mu\partial_y[(\theta_1 + \Theta\phi' + 1)\eta_4\bar{h}] - \mu\partial_y[(\theta_1 + \Theta\phi' + 1)\eta_4\eta_6\tilde{\psi}] - [(h_1 + H\phi')\partial_x + (g_1 - H_x\phi)\partial_y]\bar{h} \\ \quad + b_1\bar{u} + b_2\bar{\theta} + b_3\bar{h} + c_1\tilde{\psi} - (\mu\partial_y u_2 + U\phi'')\partial_y \bar{\theta} = 0, \\ c_v [\partial_t + (u_1 + U\phi')\partial_x + (v_1 - U_x\phi)\partial_y]\bar{\theta} - \kappa\partial_y^2 \bar{\theta} + b_4\bar{u} + b_5\bar{\theta} + b_6\bar{h} + c_2\tilde{\psi} - \mu a_1 \partial_y \bar{u} \\ \quad + [(c_v\nu - \kappa)\eta_5 - \nu(2H\phi' + \partial_y h_1 + \partial_y h_2)]\partial_y \bar{h} = 0, \\ \partial_t \bar{h} + [(u_1 + U\phi')\partial_x + (v_1 - U_x\phi)\partial_y]\bar{h} - [(h_1 + H\phi')\partial_x + (g_1 - H_x\phi)\partial_y]\bar{u} - \nu\partial_y^2 \bar{h} \\ \quad + b_7\bar{u} + b_8\bar{h} + c_3\tilde{\psi} = 0, \end{array} \right. \quad (4.20)$$

where

$$\begin{aligned}
 b_1 &\triangleq \partial_x u_2 + U_x \phi' + (g_2 - H_x \phi) \eta_4, \quad b_2 \triangleq -\mu \partial_y^2 u_2 - U \phi^{(3)}, \\
 b_3 &\triangleq \nu(\eta_4 \eta_6 + 2\partial_y \eta_4) - (g_2 - H_x \phi) \eta_6 - (\partial_x h_2 + H_x \phi') - 2\nu \partial_y \eta_4 - (\mu \partial_y u_2 + U \phi'') \eta_5, \\
 c_1 &\triangleq \partial_t \eta_4 + [(u_1 + U \phi') \partial_x + (v_1 - U_x \phi) \partial_y] \eta_4 - \nu \partial_y^2 \eta_4 + \nu(\eta_4 \eta_6^2 + \eta_4 \partial_y \eta_6 + 2\eta_6 \partial_y \eta_4 + \partial_y^2 \eta_4) \\
 &\quad - [(h_1 + H \phi') \partial_x + (g_1 - H_x \phi) \partial_y] \eta_6 - (g_2 - H_x \phi) \eta_6^2 + [\partial_x u_2 + U_x \phi' + \eta_4 (g_2 - H_x \phi)] \eta_4 \\
 &\quad - (\partial_x h_2 + H_x \phi' + 2\nu \partial_y \eta_4) \eta_6 - (\mu \partial_y u_2 + U \phi'') (\partial_y \eta_5 + \eta_5 \eta_6) - (\mu \partial_y^2 u_2 + U \phi^{(3)}) \eta_5, \\
 b_4 &\triangleq c_v [\partial_x \theta_2 + \partial_x \Theta \phi' + \eta_5 (g_2 - H_x \phi) + \eta_5 (g_2 - H_x \phi)], \quad b_5 \triangleq -\mu [(U \phi'')^2 + (\partial_y u_1)^2 + 2U \phi'' \partial_y u_1], \\
 b_6 &\triangleq (c_v \nu - \kappa) (\eta_5 \eta_6 + 2\partial_y \eta_5) - \nu \eta_6 (2H \phi' + \partial_y h_1 + \partial_y h_2) - \mu \eta_4 a_1, \\
 c_2 &\triangleq (c_v \nu - \kappa) (\eta_5 \eta_6^2 + \eta_5 \partial_y \eta_6 + \partial_y^2 \eta_5 + 2\eta_6 \partial_y \eta_5) + c_v \eta_4 [\partial_x \theta_2 + \partial_x \Theta \phi' + \eta_5 (g_2 - H_x \phi)] \\
 &\quad - \mu \eta_5 [(U \phi'')^2 + (\partial_y u_1)^2 + 2U \phi'' \partial_y u_1] - \mu (\eta_4 \eta_6 + \partial_y \eta_4) a_1 - 2c_v \nu \eta_6 \partial_y \eta_5 + c_v \eta_4 \eta_5 (g_2 - H_x \phi) \\
 &\quad + c_v (\partial_t \eta_5 + [(u_1 + U \phi') \partial_x + (v_1 - U_x \phi) \partial_y] \eta_5 - \nu \partial_y^2 \eta_5) - \nu (\eta_6^2 + \partial_y \eta_6) (2H \phi' + \partial_y h_1 + \partial_y h_2), \\
 b_7 &\triangleq \partial_x h_2 + H_x \phi' + \eta_6 (g_2 - H_x \phi), \quad b_8 \triangleq -(\partial_x u_2 + U_x \phi') - 2\nu \partial_y \eta_6 - \eta_4 (g_2 - H_x \phi), \\
 c_3 &\triangleq \partial_t \eta_6 + [(u_1 + U \phi') \partial_x + (v_1 - U_x \phi) \partial_y] \eta_6 - \nu \partial_y^2 \eta_6 - [(h_1 + H \phi') \partial_x + (g_1 - H_x \phi) \partial_y] \eta_4 \\
 &\quad - 2\nu \eta_6 \partial_y \eta_6 + (\partial_x h_2 + H_x \phi') \eta_4 - (\partial_x u_2 + U_x \phi') \eta_6.
 \end{aligned}$$

Furthermore, we can also obtain the following boundary condition

$$(\bar{u}, \bar{\theta}, \partial_y \bar{h})|_{y=0} = 0, \quad (4.21)$$

and the initial data

$$\begin{cases} \bar{u}(0, x, y) = u_{10} - u_{20} - \frac{\partial_y u_{20} + U(0, x) \phi''(y)}{h_{20} + H(0, x) \phi'(y)} \partial_y^{-1} (h_{10} - h_{20}), \\ \bar{\theta}(0, x, y) = \theta_{10} - \theta_{20} - \frac{\partial_y \theta_{20} + \Theta(0, x) \phi''(y)}{h_{20} + H(0, x) \phi'(y)} \partial_y^{-1} (h_{10} - h_{20}), \\ \bar{h}(0, x, y) = h_{10} - h_{20} - \frac{\partial_y h_{20} + H(0, x) \phi''(y)}{h_{20} + H(0, x) \phi'(y)} \partial_y^{-1} (h_{10} - h_{20}). \end{cases} \quad (4.22)$$

Furthermore, similar to [42], it is easy to deduce from (4.19) that

$$\bar{h} = (h_2 + H \phi') \partial_y \left\{ \frac{\tilde{\psi}}{h_2 + H \phi'} \right\},$$

which, together with the homogeneous boundary condition (i.e., $\tilde{\psi}|_{y=0} = 0$), yields directly

$$\tilde{\psi}(t, x, y) = (h_2(t, x, y) + H(t, x) \phi'(y)) \int_0^y \frac{\bar{h}(t, x, z)}{h_2(t, x, z) + H(t, x) \phi'(z)} dz. \quad (4.23)$$

In view of $h_2 + H \phi' \geq \delta_0$, then we applying inequality (A.7) and the representation (4.23) to get

$$\|\langle y \rangle^{-1} \tilde{\psi}(t)\|_{L^2(\Omega)} \leq 2\delta_0^{-1} \|h_2 + H \phi'\|_{L^\infty([0, T_*] \times \Omega)} \|\bar{h}(t)\|_{L^2(\Omega)}. \quad (4.24)$$

On the other hand, similar to (3.105), one can get that there exists a constant

$$C = C(T_*, \delta_0, \phi, U, \Theta, H, \|(u_1, \theta_1, h_1)\|_{\mathcal{H}_l^5}, \|(u_2, \theta_2, h_2)\|_{\mathcal{H}_l^5}) > 0,$$

such that

$$\|a_1\|_{L^\infty([0, T_*] \times \Omega)}, \|b_i\|_{L^\infty([0, T_*] \times \Omega)}, \|\langle y \rangle c_j\|_{L^\infty([0, T_*] \times \Omega)} \leq C, \quad i = 1, 2, \dots, 8, \quad j = 1, 2, 3, \quad (4.25)$$

which, together with (4.24), yields directly

$$\|(c_j \tilde{\psi})(t)\|_{L^2(\Omega)} \leq C \|\bar{h}(t)\|_{L^2(\Omega)}, \quad j = 1, 2, 3. \quad (4.26)$$

Now, we can establish the following proposition for the quantity $(\bar{u}, \bar{\theta}, \bar{h})$ that will play an important in giving the uniqueness of solution for nonlinear MHD boundary layer problem (2.6).

Proposition 4.2. *Let $(u_1, v_1, \theta_1, h_1, g_1)$ and $(u_2, v_2, \theta_2, h_2, g_2)$ be two solutions of the problem (2.6) with respect to the initial data $(u_{10}, \theta_{10}, h_{10})$ and $(u_{20}, \theta_{20}, h_{20})$ respectively, satisfying that $(u_i, \theta_i, h_i) \in \cap_{i=0}^m W^{i,\infty}(0, T_*; H_l^{m-i}(\Omega))$ for $m \geq 5, i = 1, 2$. Then, there exists a positive constant*

$$C = C(T_*, \delta_0, \phi, U, \Theta, H, \|(u_1, \theta_1, h_1)\|_{\mathcal{H}_l^5}, \|(u_2, \theta_2, h_2)\|_{\mathcal{H}_l^5}) > 0, \quad (4.27)$$

such that for the quantity given by satisfying the following differential inequality:

$$\frac{d}{dt} \|(\bar{u}, \bar{\theta}, \bar{h})(t)\|_{L^2(\Omega)}^2 + \|(\partial_y \bar{u}, \partial_y \bar{\theta}, \partial_y \bar{h})(t)\|_{L^2(\Omega)}^2 \leq C \|(\bar{u}, \bar{\theta}, \bar{h})(t)\|_{L^2(\Omega)}^2. \quad (4.28)$$

Proof. Multiplying equations (4.20)₁ and (4.20)₃ by \bar{u} and \bar{h} respectively, and integrating by part, we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (|\bar{u}|^2 + |\bar{h}|^2) dx dy + \mu \int_{\Omega} (\theta_1 + \Theta \phi' + 1) |\partial_y \bar{u}|^2 dx dy + \nu \int_{\Omega} |\partial_y \bar{h}|^2 dx dy \\ &= -\mu \int_{\Omega} (\theta_1 + \Theta \phi' + 1) (\partial_y \eta_4 \tilde{\psi} + \eta_4 \eta_6 \tilde{\psi} + \eta_4 \bar{h}) \partial_y \bar{u} dx dy - \int_{\Omega} (\mu \partial_y u_2 + U \phi'') \partial_y \bar{\theta} \cdot \bar{u} dx dy \\ & \quad - \int_{\Omega} (b_1 \bar{u} + b_2 \bar{\theta} + b_3 \bar{h} + c_1 \tilde{\psi}) \cdot \bar{u} dx dy - \int_{\Omega} (b_7 \bar{u} + b_8 \bar{h} + c_3 \tilde{\psi}) \cdot \bar{h} dx dy. \end{aligned}$$

By using lower bound estimate for temperature (3.5), the estimates (4.25)-(4.26) and Cauchy inequality, it is easy to check that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (|\bar{u}|^2 + |\bar{h}|^2) dx dy + \mu \int_{\Omega} |\partial_y \bar{u}|^2 dx dy + \nu \int_{\Omega} |\partial_y \bar{h}|^2 dx dy \\ & \leq \frac{\mu}{2} \int_{\Omega} |\partial_y \bar{u}|^2 dx dy + \frac{\kappa}{2} \int_{\Omega} |\partial_y \bar{\theta}|^2 dx dy + C \int_{\Omega} |(\bar{u}, \bar{\theta}, \bar{h})|^2 dx dy. \end{aligned} \quad (4.29)$$

Multiplying the equation (4.20)₂ by $\bar{\theta}$ and integrating by part, one arrives at directly

$$\frac{c_v}{2} \frac{d}{dt} \int_{\Omega} |\bar{\theta}|^2 dx dy + \kappa \int_{\Omega} |\partial_y \bar{\theta}|^2 dx dy = \mu \int_{\Omega} a_1 \partial_y \bar{u} \cdot \bar{\theta} dx dy - \int_{\Omega} (b_4 \bar{u} + b_5 \bar{\theta} + b_6 \bar{h} + c_2 \tilde{\psi}) \cdot \bar{\theta} dx dy.$$

Then, we apply the Cauchy inequality and the estimates (4.25)-(4.26) to get that

$$\frac{c_v}{2} \frac{d}{dt} \int_{\Omega} |\bar{\theta}|^2 dx dy + \kappa \int_{\Omega} |\partial_y \bar{\theta}|^2 dx dy \leq \frac{\mu}{2} \int_{\Omega} |\partial_y \bar{u}|^2 dx dy + C \int_{\Omega} |(\bar{u}, \bar{\theta}, \bar{h})|^2 dx dy,$$

which, together with the inequality (4.29), yields the estimate (4.28). Therefore, we complete the proof of Proposition 4.2. \square

Proof of Uniqueness. Indeed, if the initial data satisfying $(u_{10}, \theta_{10}, h_{10}) = (u_{20}, \theta_{20}, h_{20})$, then we deduce from the representation (4.22) that $(\bar{u}, \bar{\theta}, \bar{h})$ admits the zero initial data $(\bar{u}, \bar{\theta}, \bar{h})|_{t=0} = 0$. Then, we apply the Grönwall inequality to (4.28) to get that $(\bar{u}, \bar{\theta}, \bar{h}) = 0$. Putting $\bar{h} \equiv 0$ into the representation (4.23), one get that $\tilde{\psi} \equiv 0$. By direct calculation, one arrives at directly

$$(u_1, \theta_1, h_1) - (u_2, \theta_2, h_2) = (\bar{u}, \bar{\theta}, \bar{h}) + (\eta_4, \eta_5, \eta_6) \psi = 0,$$

which yields that $(u_1, \theta_1, h_1) \equiv (u_2, \theta_2, h_2)$. Finally, in view of the relation

$$v_i = -\partial_y^{-1} \partial_x u_i, \quad g_i = -\partial_y^{-1} \partial_x h_i, \quad i = 1, 2,$$

we find the uniqueness of solution of the nonlinear MHD boundary layer problem (2.6). \square

A Calculus Inequalities

In this appendix, we will introduce some basic inequality that be used frequently in this paper. For the proof in detail, the interested readers can refer to [42].

Lemma A.1. *For proper functions f, g, h , the following holds.*

(i) If $\lim_{y \rightarrow +\infty} (fg)(x, y) = 0$, then

$$\left| \int_{\mathbb{T}_x} (fg)|_{y=0} \right| \leq \|\partial_y f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|\partial_y g\|_{L^2(\Omega)}. \quad (\text{A.1})$$

In particular, if $\lim_{y \rightarrow +\infty} f(x, y) = 0$, then

$$\|f|_{y=0}\|_{L^2(\mathbb{T}_x)} \leq \sqrt{2} \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\partial_y f\|_{L^2(\Omega)}^{\frac{1}{2}}. \quad (\text{A.2})$$

(ii) If $l \in \mathbb{R}$ and an integer $m \geq 3$, any $\alpha = (\beta, k) \in \mathbb{N}^3$, $\tilde{\alpha} = (\tilde{\beta}, \tilde{k}) \in \mathbb{N}^3$ with $|\alpha| + |\tilde{\alpha}| \leq m$,

$$\|(D^\alpha f \cdot D^{\tilde{\alpha}} g)(t, \cdot)\|_{L^2_{l+k+\tilde{k}}(\Omega)} \leq C \|f(t)\|_{\mathcal{H}^m_{l_1}} \|g(t)\|_{\mathcal{H}^m_{l_2}}, \text{ for all } l_1, l_2 \in \mathbb{R}, \quad l_1 + l_2 = l. \quad (\text{A.3})$$

(iii) For any $\lambda > \frac{1}{2}$, $\tilde{\lambda} > 0$,

$$\|\langle y \rangle^{-\lambda} (\partial_y^{-1} f)(y)\|_{L^2_y(\mathbb{R}_+)} \leq \frac{2}{2\lambda - 1} \|\langle y \rangle^{1-\lambda} f(y)\|_{L^2_y(\mathbb{R}_+)} \quad (\text{A.4})$$

and

$$\|\langle y \rangle^{-\tilde{\lambda}} (\partial_y^{-1} f)(y)\|_{L^\infty_y(\mathbb{R}_+)} \leq \frac{1}{\tilde{\lambda}} \|\langle y \rangle^{1-\tilde{\lambda}} f(y)\|_{L^\infty_y(\mathbb{R}_+)} \quad (\text{A.5})$$

and then, for $l \in \mathbb{R}$, an integer $m \geq 3$, and any $\alpha = (\beta, k) \in \mathbb{N}^3$, $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2) \in \mathbb{N}^2$ with $|\alpha| + |\tilde{\beta}| \leq m$,

$$\|(D^\alpha g \cdot \partial_\tau^{\tilde{\beta}} \partial_y^{-1} h)(t, \cdot)\|_{L^2_{l+k}(\Omega)} \leq C \|g(t)\|_{\mathcal{H}^m_{l+\lambda}} \|h(t)\|_{\mathcal{H}^m_{1-\lambda}}. \quad (\text{A.6})$$

In particular, for $\lambda = 1$,

$$\|\langle y \rangle^{-1} (\partial_y^{-1} f)(y)\|_{L^2_y(\mathbb{R}_+)} \leq 2 \|f\|_{L^2_y(\mathbb{R}_+)}, \quad (\text{A.7})$$

and

$$\|(D^\alpha g \cdot \partial_\tau^{\tilde{\beta}} \partial_y^{-1} h)(t, \cdot)\|_{L^2_{l+k}(\Omega)} \leq C \|g(t)\|_{\mathcal{H}^m_{l+1}} \|h(t)\|_{\mathcal{H}^m_0}. \quad (\text{A.8})$$

B Almost Equivalence of Weighted Norms

In this subsection, we give the almost equivalence in L^2_l -norm between $\partial_\tau^\beta(u, \theta, h)$ and the quantity $(u_\beta, \theta_\beta, h_\beta)$ defined in (3.94).

Lemma B.1. *If the smooth function (u, θ, h) satisfies the nonlinear problem (2.6) in $[0, T]$, and the assumption condition (3.1) holds on, then for any $t \in [0, T]$, any real number $l \geq 0$, an integer $m \geq 3$ and the quantity $(u_\beta, \theta_\beta, h_\beta)$ with $|\beta| = m$ defined by (3.94), we have the following relations*

$$M(t)^{-1} \|\partial_\tau^\beta(u, \theta, h)(t)\|_{L^2_l(\Omega)} \leq \|(u_\beta, \theta_\beta, h_\beta)(t)\|_{L^2_l(\Omega)} \leq M(t) \|\partial_\tau^\beta(u, \theta, h)(t)\|_{L^2_l(\Omega)} \quad (\text{B.1})$$

and

$$\|\partial_y \partial_\tau^\beta(u, \theta, h)(t)\|_{L^2_l(\Omega)} \leq \|\partial_y(u_\beta, \theta_\beta, h_\beta)(t)\|_{L^2_l(\Omega)} + M(t) \|h_\beta(t)\|_{L^2_l(\Omega)}, \quad (\text{B.2})$$

where the function $M(t)$ is defined by (3.97).

Proof. Firstly, by the definition of (3.94), we find

$$\begin{aligned}\|\theta_\beta\|_{L_t^2(\Omega)} &\leq \|\partial_\tau^\beta \theta\|_{L_t^2(\Omega)} + \|\langle y \rangle^{l+1} \eta_2\|_{L^\infty(\Omega)} \|\langle y \rangle^{-1} \partial_\tau^\beta \psi\|_{L^2(\Omega)} \\ &\leq \|\partial_\tau^\beta \theta\|_{L_t^2(\Omega)} + 2\delta_0^{-1} (\|\langle y \rangle^{l+1} \partial_y \theta\|_{L^\infty(\Omega)} + C \|\Theta\|_{L^\infty(\mathbb{T}_x)}) \|\partial_\tau^\beta h\|_{L^2(\Omega)} \\ &\leq M(t) \|\partial_\tau^\beta(\theta, h)\|_{L_t^2(\Omega)}.\end{aligned}\tag{B.3}$$

Similarly, we obtain the following estimate(or see Liu et al.[42])

$$\|(u_\beta, h_\beta)\|_{L_t^2(\Omega)} \leq M(t) \|\partial_\tau^\beta(u, h)\|_{L_t^2(\Omega)}.\tag{B.4}$$

On the other hand, in view of the definition of h_β in (3.94), one attains

$$h_\beta = \partial_\tau^\beta h - \eta_3 \partial_\tau^\beta \psi = (h + H\phi') \partial_y \left\{ \frac{\partial_\tau^\beta \psi}{h + H\phi'} \right\},$$

which, together with the boundary condition $\psi|_{y=0} = 0$, implies directly

$$\partial_\tau^\beta \psi = (h + H\phi') \int_0^y \frac{h_\beta}{h + H\phi'} d\tilde{y}.\tag{B.5}$$

Substituting (B.5) into (3.94), one arrives at

$$\partial_\tau^\beta \theta = \theta_\beta + (\partial_y \theta + \Theta \phi'') \int_0^y \frac{h_\beta}{h + H\phi'} d\tilde{y}.\tag{B.6}$$

Then, it is easy to check that

$$\begin{aligned}\|\partial_\tau^\beta \theta\|_{L_t^2(\Omega)} &\leq \|\theta_\beta\|_{L_t^2(\Omega)} + \|\langle y \rangle^{l+1} (\partial_y \theta + \Theta \phi'')\|_{L^\infty(\Omega)} \|\langle y \rangle^{-1} \int_0^y \frac{h_\beta}{h + H\phi'} d\tilde{y}\|_{L^2(\Omega)} \\ &\leq \|\theta_\beta\|_{L_t^2(\Omega)} + 2\delta_0^{-1} (\|\langle y \rangle^{l+1} \partial_y \theta\|_{L^\infty(\Omega)} + C \|\Theta\|_{L^\infty(\mathbb{T}_x)}) \|h_\beta\|_{L^2(\Omega)} \\ &\leq M(t) \|(\theta_\beta, h_\beta)\|_{L_t^2(\Omega)}.\end{aligned}\tag{B.7}$$

Furthermore, taking y derivative to both handside of representation (B.6), we find

$$\partial_y \partial_\tau^\beta \theta = \partial_y \theta_\beta + (\partial_y^2 \theta + \Theta \phi^{(3)}) \int_0^y \frac{h_\beta}{h + H\phi'} d\tilde{y} + \frac{h_\beta (\partial_y \theta + \Theta \phi'')}{h + H\phi'}.\tag{B.8}$$

Hence, it is easy to deduce that

$$\begin{aligned}\|\partial_y \partial_\tau^\beta \theta\|_{L_t^2(\Omega)} &\leq \|\partial_y \theta_\beta\|_{L_t^2(\Omega)} + \|\langle y \rangle^{l+1} (\partial_y^2 \theta + \Theta \phi^{(3)})\|_{L^\infty(\Omega)} \|\langle y \rangle^{-1} \int_0^y \frac{h_\beta}{h + H\phi'} d\tau\|_{L^2(\Omega)} \\ &\quad + \delta_0^{-1} \|\partial_y \theta + \Theta \phi''\|_{L^\infty(\Omega)} \|h_\beta\|_{L_t^2(\Omega)} \\ &\leq \|\partial_y \theta_\beta\|_{L_t^2(\Omega)} + M(t) \|h_\beta\|_{L_t^2(\Omega)}.\end{aligned}\tag{B.9}$$

Similarly, we obtain the following estimates(or see [42])

$$\|\partial_\tau^\beta(u, h)\|_{L_t^2(\Omega)} \leq M(t) \|(u_\beta, h_\beta)\|_{L_t^2(\Omega)},\tag{B.10}$$

and

$$\|\partial_y \partial_\tau^\beta(u, h)\|_{L_t^2(\Omega)} \leq \|\partial_y(u_\beta, h_\beta)\|_{L_t^2(\Omega)} + M(t) \|h_\beta\|_{L_t^2(\Omega)}.\tag{B.11}$$

Therefore, we complete the proof of the Lemma B.1. \square

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